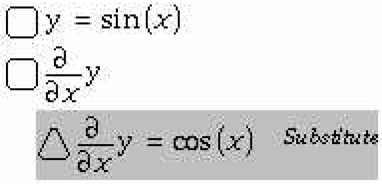
Differential Equations

Testing Solutions

• Using Case Theories to test more than one solution

Finding the derivative of a function was the subject of the chapter on differential calculus. See pages 156 and 165. When you reverse this process, you are solving a differential equation. This section introduces you to differential equations by showing how you can use Case Theories to test potential solutions.

When you substitute the first Prop below into the Derivative Op, you generate a new function along with a derivative. Not only have you found the derivative of the first function, you have created a differential equation.



Equation containing function

☐ $y = \sin(x)$

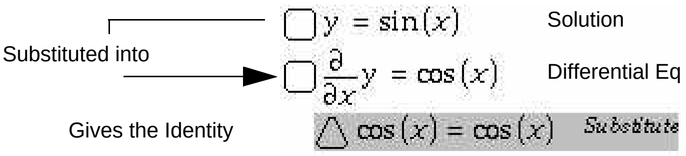
☐ $\frac{\partial}{\partial x} y$

Differential Equation

△ $\frac{\partial}{\partial x} y = \cos(x)$ Substitute

You can find the solution to the differential equation above by reversing the process.

A solution to the differential equation is the function $y = f(x)$ that, when substituted into the associated differential equation, reduces that equation to an identity. For example, you would reduce the differential equation above to an identity by substituting the original function (the solution) into it.



☐ $y = \sin(x)$ Solution

Substituted into

☐ $\frac{\partial}{\partial x} y = \cos(x)$ Differential Eq

Gives the Identity

△ $\cos(x) = \cos(x)$ Substitute

Therefore, $y = \sin(x)$ is a solution to the differential equation $dy/dx = \cos(x)$.

Explicit Solution

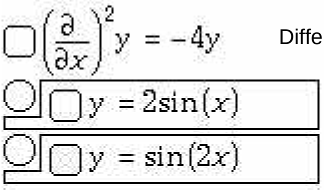
A solution that you place in the form $y = f(x)$ is called an explicit solution. The unknown (dependent) variable, ***y***, is a function of the known (independent) variable, ***x***. In the example above, $y = \sin(x)$ is an explicit solution to the differential equation $dy/dx = \cos(x)$.

Implicit Solution

When you cannot isolate the dependent variable, as you could with the equation above, the solution is called implicit. The solution is in the form $f(x, y) = 0$. You will find *MathView* very useful in determining whether a function is a solution to a given differential equation. Below is an example.

- Set up a *MathView* notebook with the differential equation in its own Prop, followed by two potential solutions in separate Case Theories. Use **Notebook ▶ Insert ▶ Case Theory**.

- Select each potential solution and substitute into the differential equation with the Hand and **Expand** the result.



$$\left(\frac{\partial}{\partial x}\right)^2 y = -4y$$

Differential Equation

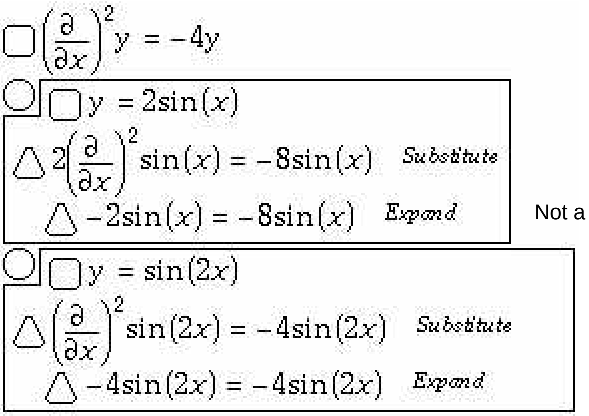
$$y = 2\sin(x)$$

Potential Solution #1

$$y = \sin(2x)$$

Potential Solution #2

MathView brings the differential equation into each Case Theory, allowing each solution to be tested separately.



$$\left(\frac{\partial}{\partial x}\right)^2 y = -4y$$

$$y = 2\sin(x)$$

$$2\left(\frac{\partial}{\partial x}\right)^2 \sin(x) = -8\sin(x) \quad \textit{Substitute}$$

$$-2\sin(x) = -8\sin(x) \quad \textit{Expand}$$

Not a Solution

$$y = \sin(2x)$$

$$\left(\frac{\partial}{\partial x}\right)^2 \sin(2x) = -4\sin(2x) \quad \textit{Substitute}$$

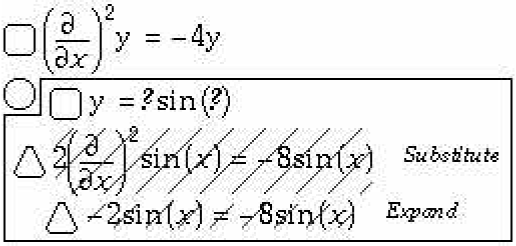
$$-4\sin(2x) = -4\sin(2x) \quad \textit{Expand}$$

A Solution

To toggle on ReManipulate choose **Always ReManipulate** under the **Manipulate** menu.

You can use one Case Theory to test several potential solutions. Before you start, make sure **Always ReManipulate** is on.

- Using the first Case Theory above, change the potential solution $2\sin(x)$ to the correct solution $\sin(2x)$.



$$\left(\frac{\partial}{\partial x}\right)^2 y = -4y$$

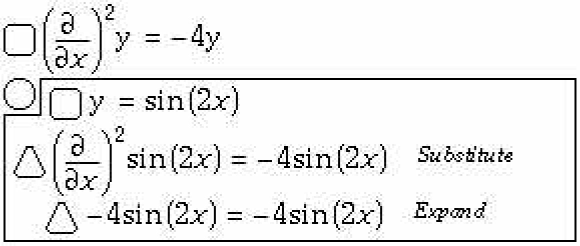
$$y = ?\sin(?)$$

$$\cancel{2\left(\frac{\partial}{\partial x}\right)^2 \sin(x) = -8\sin(x)} \quad \textit{Substitute}$$

$$\cancel{-2\sin(x) = -8\sin(x)} \quad \textit{Expand}$$

Original Solution in the process of changing.

- Below is the result after you have altered the solution.



$$\left(\frac{\partial}{\partial x}\right)^2 y = -4y$$

$$y = \sin(2x)$$

$$\left(\frac{\partial}{\partial x}\right)^2 \sin(2x) = -4\sin(2x) \quad \textit{Substitute}$$

$$-4\sin(2x) = -4\sin(2x) \quad \textit{Expand}$$

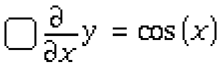
Simple Integration with Initial Value

• Using the Auto Simplify option to generate constants of integration

When you differentiate a function, the resulting conclusion has a new function along with a derivative in it. Reversing this procedure, by integrating the conclusion, produces the original equation along with a constant of integration. Therefore, every time you integrate, you are solving a differential equation. In this section, you use the **Apply** manipulation to help solve a simple differential equation.

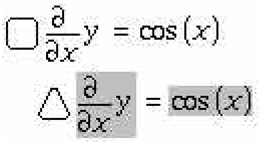
You call the result you obtain in the following example the General Solution to the differential equation. If given an initial value for ***x*** and $f(x)$, you can find a Particular Solution. Below is an example.

- Input the following differential equation.



$$\square\ \frac{\partial}{\partial x}y = \cos(x)$$

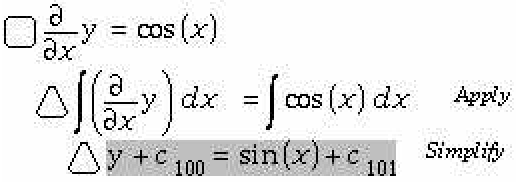
- Select the whole equation, by clicking on the equal sign once, then choose **Apply**.



$$\square\ \frac{\partial}{\partial x}y = \cos(x)$$

$$\triangle\ \frac{\partial}{\partial x}y = \cos(x)$$

- Click on the Indefinite Integral Op on the palette and type the independent variable (***x*** in this case). Select the equation and **Simplify** with **Auto Casing** turned on.



$$\square\ \frac{\partial}{\partial x}y = \cos(x)$$

$$\triangle\ \int\left(\frac{\partial}{\partial x}y\right)dx = \int\cos(x)\,dx$$

$$\triangle\ y + c_{100} = \sin(x) + c_{101}$$

- **Isolate** ***y*** for the General Solution.

$$\triangle\ y = \sin(x) + c_{101} - c_{100}$$

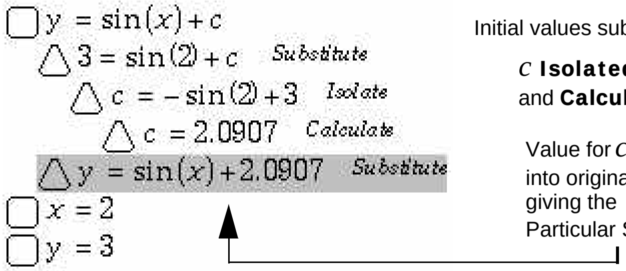
Essential Constants

You would normally combine the two constants of integration, in the example above, into one. This demonstrates the concept of essential constants. Because there are two integrations, two constants are generated. Since these two constants can be combined into one, without effecting the solution, one or the other is non-essential, and you can be eliminated.

- Select the two constants and type a ***c*** to replace both with a new, unlabeled ***c***. This action also generates a new Assumption.

Given an Initial Value, IV, you can find the Particular Solution by substituting those values into the General Solution, solving for ***c***, and substituting ***c*** back into the General Solution.

- Use an initial value of $(2, 3)$, that is $(f(2) = 3)$.
- Select both the ***x*** and ***y*** equations and substitute into the first Prop.
- **Isolate *c*.**
- Select the resulting equation and **Calculate**.
- You arrive at the final solution (Particular Solution) by substituting the equation for ***c*** into the original equation (General Solution) in the first Prop.



☐ $y = \sin(x) + c$

△ $3 = \sin(2) + c$ Substitute

△ $c = -\sin(2) + 3$ Isolate

△ $c = 2.0907$ Calculate

△ $y = \sin(x) + 2.0907$ Substitute

☐ $x = 2$

☐ $y = 3$

Initial values substituted in

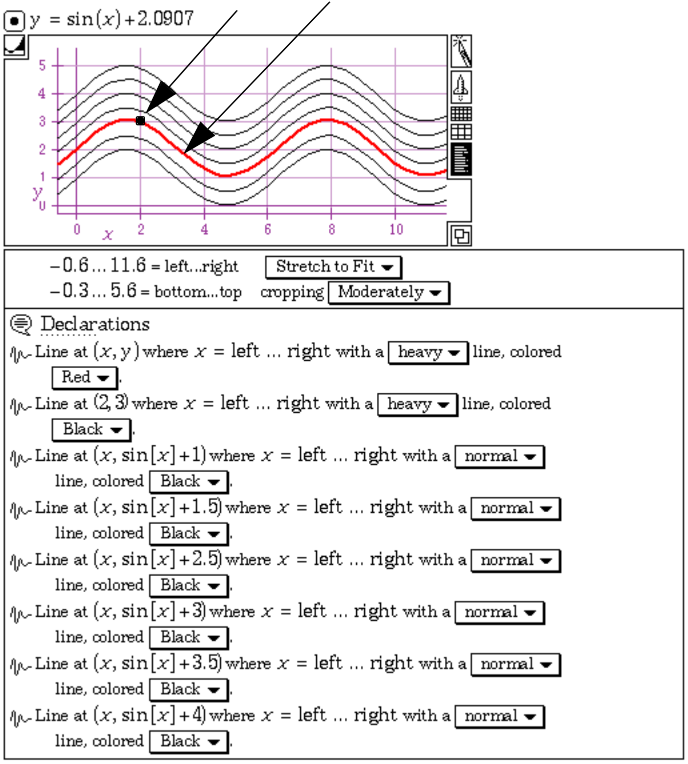
c **Isolated**
and **Calculated**

Value for *c* substituted
into original equation
giving the
Particular Solution

Below is a Graph Theory containing several solution curves, including the Particular Solution with the Initial Value (2,3).

IV

Particular Solution Curve



● $y = \sin(x) + 2.0907$

$-0.6 \ldots 11.6$ = left...right Stretch to Fit

$-0.3 \ldots 5.6$ = bottom...top cropping Moderately

Line at (x, y) where x = left ... right with a heavy line, colored Red.

Line at $(2, 3)$ where x = left ... right with a heavy line, colored Black.

Line at $(x, \sin[x]+1)$ where x = left ... right with a normal line, colored Black.

Line at $(x, \sin[x]+1.5)$ where x = left ... right with a normal line, colored Black.

Line at $(x, \sin[x]+2.5)$ where x = left ... right with a normal line, colored Black.

Line at $(x, \sin[x]+3)$ where x = left ... right with a normal line, colored Black.

Line at $(x, \sin[x]+3.5)$ where x = left ... right with a normal line, colored Black.

Line at $(x, \sin[x]+4)$ where x = left ... right with a normal line, colored Black.

Separation of Variables

- Entering derivatives using the differential operator
- Using the Apply manipulation to help solve differential equations

You can algebraically manipulate an equation in the form of $dy/dx = g(x)/h(y)$ into the following form.

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \rightarrow h(y)\,dy = g(x)\,dx$$

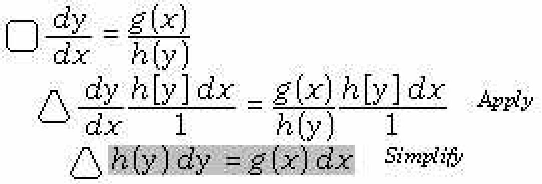
Notice how all the ***x***s are together on one side of the equation and all the ***y***s are together on the other. Use the old algebraic method, cross multiplying, to put the equation into this form. In *MathView*, you manipulate with the Hand, use the rules of algebra, or re-input in the new form. In the first example, you will use the algebraic method.

You must enter the differential in the following form for the operation to work; the Partial Derivative Op will not work.

- Enter **`(d*y)/(d*x)`** [esc] **`= (g(x))/(h(y))`** or **`/d*x`** [tab] **`d*y Esc = / g(x`** [tab] **`h(y`**
- Select the equation and choose **`Apply`** (click on the palette image).
- Type **`* (h(y)*d*x)/1`**
- Select by clicking on the equal sign and **`Simplify`**.

Both ***g*** and ***h*** must be User Defined Functions in this example.

You may also perform the operation on the right by selecting ***h***(***y***) and dragging it to the LHS. Then **`Move Over`** 1/***dx*** to the RHS.



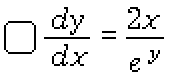
$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$\frac{dy}{dx}\,\frac{h[y]\,dx}{1} = \frac{g(x)}{h(y)}\,\frac{h[y]\,dx}{1} \quad \text{Apply}$$

$$h(y)\,dy = g(x)\,dx \quad \text{Simplify}$$

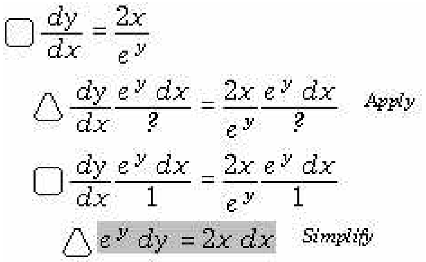
Now that the expression is in differential form; you can integrate each side for the solution.

- Now try entering the following equation using your favorite method.



$$\frac{dy}{dx} = \frac{2x}{e^{y}}$$

- Using the same method you used above, place the differential equation into a separated form. Remember to separate the differential operators from their variables with a space, or a multiplication (both **`d*x`** and **`d · x`** produces ***dx***).



$$\frac{dy}{dx} = \frac{2x}{e^{y}}$$

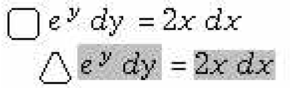
$$\frac{dy}{dx}\,\frac{e^{y}\,dx}{?} = \frac{2x}{e^{y}}\,\frac{e^{y}\,dx}{?}$$

$$\frac{dy}{dx}\,\frac{e^{y}\,dx}{1} = \frac{2x}{e^{y}}\,\frac{e^{y}\,dx}{1}$$

$$e^{y}\,dy = 2x\,dx$$

Now you can integrate. Since the differentials are already in the problem, you apply the Integral Op using the keyboard method (shift – 4).

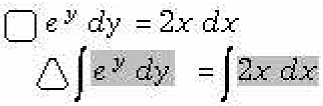
- Select the equation and choose **Apply**. Both sides highlight, ready for you to perform an operation on them. Rather than clicking on the Indefinite Integral Op on the Palette (which would add a second differential operator), enter the command key equivalent, $.



$$e^{y}\,dy = 2x\,dx$$

$$e^{y}\,dy = 2x\,dx$$

Select the equation and click on Apply



$$e^{y}\,dy = 2x\,dx$$

$$\int e^{y}\,dy = \int 2x\,dx$$

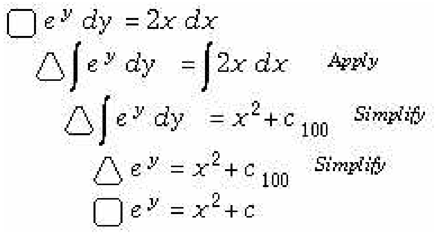
After the Integral is activated with a Shift - 4

- Select this new equation by clicking on the equal sign. Once selected, perform a **Simplify**. If **Auto Casing** is on, *MathView* generates two constants.

You can save a step by simplifying one side of the equation with **Auto Casing** turned on and the other side with it turned off. In either case, you can remove the subscript which remains after you have eliminated all non-essential constants.

The disadvantage of removing the subscript is that when *MathView* generates the new Assumption Prop, you loose the dynamic link. You will not be able to change the original problem and have it ReManipulate the solution. It may be better to wait until the very end to do this operation or just leave the subscript attached, if you plan on testing different initial values.

- Select the number and delete it; or just select the whole constant, subscript and all, and type a new ***c***. A new Prop will be generated with the subscript gone.



$$e^{y}\,dy = 2x\,dx$$

$$\int e^{y}\,dy = \int 2x\,dx$$

$$\int e^{y}\,dy = x^{2}+c_{100}$$

$$e^{y} = x^{2}+c_{100}$$

$$e^{y} = x^{2}+c$$

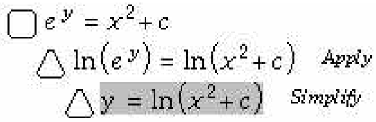
RHS selected and Simplified with Auto Casing on

LHS selected and Simplified with Auto Casing off

Subscript removed

You solve for the dependent variable ***y*** by performing an **Isolate** or by applying the Natural Log to each side.

- Select the equation by clicking on the equal sign; **Apply** the **Ln** function by clicking on its Palette image; select the resulting equation and **Simplify**.



$$e^{y} = x^{2}+c$$

$$\ln\left(e^{y}\right) = \ln\left(x^{2}+c\right)$$

$$y = \ln\left(x^{2}+c\right)$$

First Order Linear Equations

- Using the Independence Declaration
- Using Auto Casing

The general linear differential equation takes the following form.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_n(x)y = g(x)$$

Note two items in this equation. First, the derivatives and the dependent variable (***y***) are of the first degree (to the first power). Second, the coefficients depend only on the independent variable (***x***, in this case). In the context of differential equations, a coefficient can be a constant or a function of the independent variable (the dependent variable ***y*** is not multiplied onto any derivative). In addition, although not apparent from the equation above, the *dependent* variable is not the argument of a transcendental function. The independent variable can be an argument, but the dependent variable cannot.

A first order linear differential equation takes the following form.

$$a_1(x)\frac{\partial y}{\partial x} + a_0(x)y = g(x)$$

Note that both coefficients and the function ***g*** depend on the independent variable ***x*** alone and not ***y***. If you divide both sides by the coefficient of the derivative, $a_1(x)$, you place the equation into standard form.

$$\frac{a_1(x)}{a_1(x)}\frac{\partial y}{\partial x} + \frac{a_0(x)y}{a_1(x)} = \frac{g(x)}{a_1(x)}$$

The two new functions ***P*** and ***Q***, below, merely replace the two functions of ***x*** that you have created by the division. Since they are both functions of ***x***, you can replace them with new functions of ***x***, which may or may not represent a quotient in a problem.

$$\frac{\partial y}{\partial x} + P(x)y = Q(x)$$

You solve differential equations in this form by using the following integrating factor.

Integrating Factor $$u = e^{\int P(x)dx}$$

You can find the solution by placing a linear equation in the standard form $dy/dx + P(x)y = Q(x)$ and applying this integrating factor to both sides. Below is the standard formula.

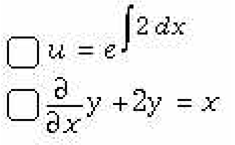
$$uy = \int uQ(x)dx + c$$

An example demonstrates how you use *MathView* to find these solutions.

- Solve the following equation.

$$\frac{\partial}{\partial x}y + 2y = x$$

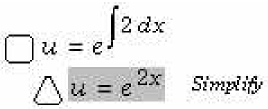
- Enter the integrating factor, defining ***u*** as a User Defined Variable. In a second Prop, enter the differential equation.



☐ $u = e^{\int 2\,dx}$

☐ $\frac{\partial}{\partial x}y + 2y = x$

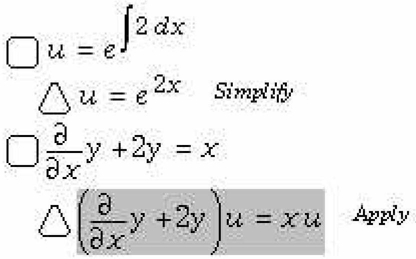
- **Simplify** the integrating factor with **Auto Casing** turned off.



☐ $u = e^{\int 2\,dx}$

△ $u = e^{2x}$ *Simplify*

- **Apply** the integrating factor to the differential equation (multiply it to both sides using a * and not ˙).



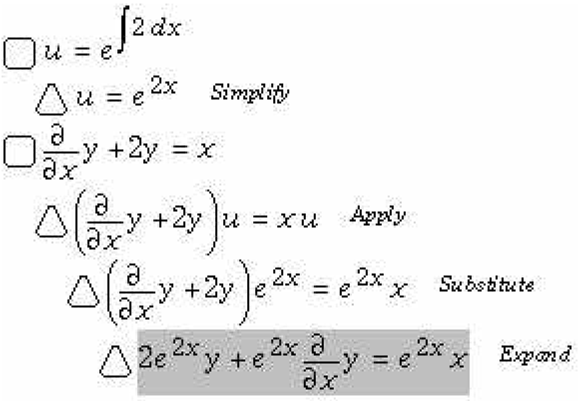
☐ $u = e^{\int 2\,dx}$

△ $u = e^{2x}$ *Simplify*

☐ $\frac{\partial}{\partial x}y + 2y = x$

△ $\left(\frac{\partial}{\partial x}y + 2y\right)u = x\,u$ *Apply*

- Substitute the integrating factor into the last conclusion with the Hand. **Expand** the resulting equation.



☐ $u = e^{\int 2\,dx}$

△ $u = e^{2x}$ *Simplify*

☐ $\frac{\partial}{\partial x}y + 2y = x$

△ $\left(\frac{\partial}{\partial x}y + 2y\right)u = x\,u$ *Apply*

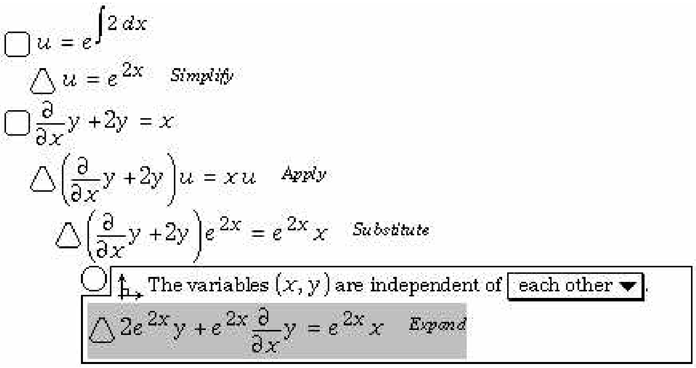
△ $\left(\frac{\partial}{\partial x}y + 2y\right)e^{2x} = e^{2x}x$ *Substitute*

△ $2e^{2x}y + e^{2x}\frac{\partial}{\partial x}y = e^{2x}x$ *Expand*

- While the last Prop is still selected, enclose it in a Case Theory by selecting **Notebook ► Insert ► Case Theory**. Insert an Independence Declaration declaring ***x*** and ***y*** independent of each other. See page 68. Move the

You add an Independence Declaration inside a Case Theory now because it is necessary for the integration to follow, but would have given the wrong answer in the substitution in the Prop just before the Case Theory (***dy/dx*** would have gone to zero).

differential equation Prop, with the Hand, to below the Independence Declaration.



☐ $u = e^{\int 2\,dx}$

△ $u = e^{2x}$ Simplify

☐ $\frac{\partial}{\partial x}y + 2y = x$

△ $\left(\frac{\partial}{\partial x}y + 2y\right)u = xu$ Apply

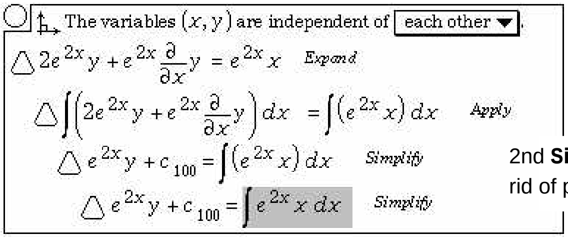
△ $\left(\frac{\partial}{\partial x}y + 2y\right)e^{2x} = e^{2x}x$ Substitute

○ The variables (x, y) are independent of each other ▾.

△ $2e^{2x}y + e^{2x}\frac{\partial}{\partial x}y = e^{2x}x$ Expand

- **Apply** the Integral Op to the last equation and **Simplify** the LHS (you should have **Auto Casing** on at this point to generate the constant). **Simplify** the RHS with **Auto Casing** off.

Notice how you turn Auto Casing on and off depending upon the circumstance.



○ The variables (x, y) are independent of each other ▾.

△ $2e^{2x}y + e^{2x}\frac{\partial}{\partial x}y = e^{2x}x$ Expand

△ $\int\left(2e^{2x}y + e^{2x}\frac{\partial}{\partial x}y\right)dx = \int\left(e^{2x}x\right)dx$ Apply

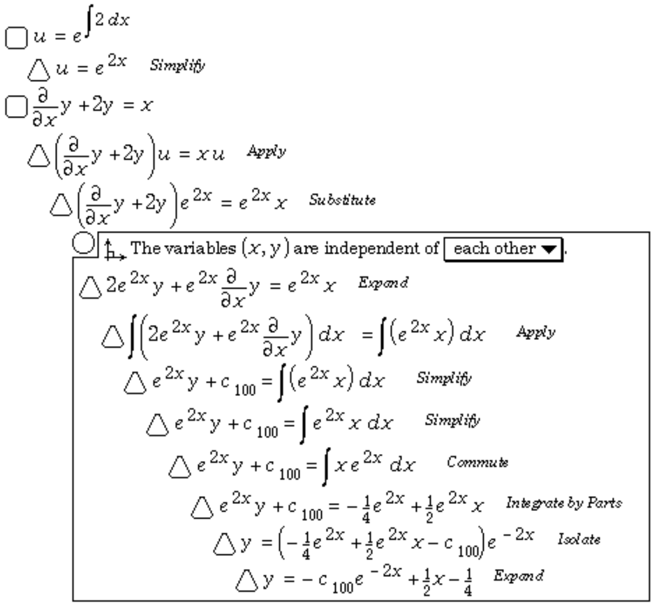
△ $e^{2x}y + c_{100} = \int\left(e^{2x}x\right)dx$ Simplify

△ $e^{2x}y + c_{100} = \int e^{2x}x\,dx$ Simplify

2nd **Simplify** gets rid of parenthesis

- *MathView* is unable to solve the RHS with a simplify manipulation. You can solve by using **Integration by Parts** on the RHS after commuting the integrand with the Hand. Make sure you turn **Auto Casing** off at this time. **Isolate *y*** and **Expand** the RHS of the resulting equation for the solution. Below is the entire Theory.

See page 174 for a review on Integration by Parts.



☐ $u = e^{\int 2\,dx}$

△ $u = e^{2x}$ Simplify

☐ $\frac{\partial}{\partial x}y + 2y = x$

△ $\left(\frac{\partial}{\partial x}y + 2y\right)u = xu$ Apply

△ $\left(\frac{\partial}{\partial x}y + 2y\right)e^{2x} = e^{2x}x$ Substitute

○ The variables (x, y) are independent of each other ▾.

△ $2e^{2x}y + e^{2x}\frac{\partial}{\partial x}y = e^{2x}x$ Expand

△ $\int\left(2e^{2x}y + e^{2x}\frac{\partial}{\partial x}y\right)dx = \int\left(e^{2x}x\right)dx$ Apply

△ $e^{2x}y + c_{100} = \int\left(e^{2x}x\right)dx$ Simplify

△ $e^{2x}y + c_{100} = \int e^{2x}x\,dx$ Simplify

△ $e^{2x}y + c_{100} = \int xe^{2x}\,dx$ Commute

△ $e^{2x}y + c_{100} = -\frac{1}{4}e^{2x} + \frac{1}{2}e^{2x}x$ Integrate by Parts

△ $y = \left(-\frac{1}{4}e^{2x} + \frac{1}{2}e^{2x}x - c_{100}\right)e^{-2x}$ Isolate

△ $y = -c_{100}e^{-2x} + \frac{1}{2}x - \frac{1}{4}$ Expand

Numerical Methods

- Euler and Runge-Kutta tables
- Scatter plots

You can use *MathView* to solve differential equations using two numerical methods, Euler and Runge-Kutta. *MathView* stores the solutions you obtain using these methods in a table which you can then examine and plot just like other tables. This section introduces these features by showing you how to solve two simple first order equations.

Euler's Method

Although it is not very accurate, Euler's method is simple and understanding it is basic to the understanding of other more sophisticated methods. The next example uses *MathView* to compare Euler's Method with an actual solution.

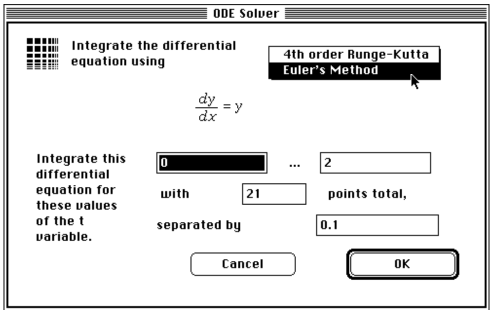
- Input the following differential equation.

MathView allows you to input the equations in either form shown to the right. The choice is up to you.

$$\square\ \frac{dy}{dx} = y \qquad \text{or} \qquad \square\ \frac{\partial}{\partial x} y = y$$

You can construct the derivative (on the left above), or you can use *MathView*'s Partial Derivative Op (right).

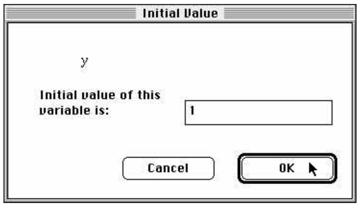
- Select the equation by clicking on the equal sign and choose **Integrate Differential Equation...** under **Table** in the **Manipulate** menu. Select Euler, rather than Runge-Kutta, by clicking on the latter and moving the mouse down to Euler's Method.



Integrate the differential equation using Euler's Method

$\frac{dy}{dx} = y$

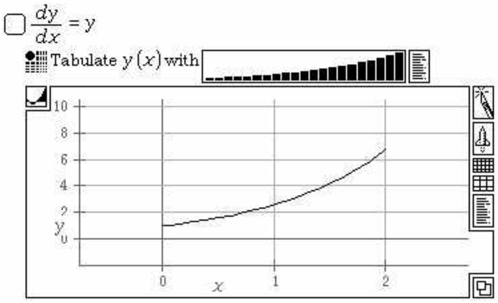
Integrate this differential equation for these values of the t variable. 0 ... 2 with 21 points total, separated by 0.1



y

Initial value of this variable is: 1

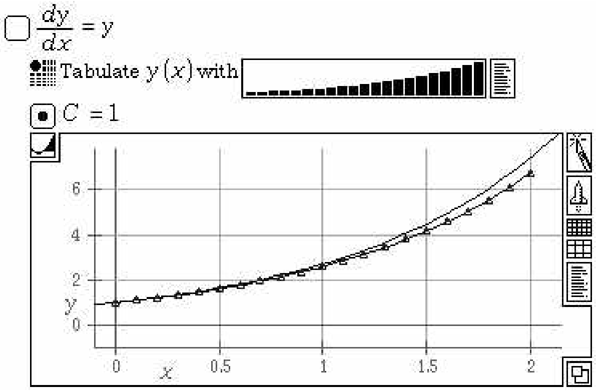
- Input the information above and click **OK**. Set the Initial Value to **1**. The table generates with 21 points according to Euler's method.
- Select the table by clicking on its leading icon and plot the points by choosing **Linear** under the **Graph** menu. Below is the table and the resulting graph theory (zoomed out).



☐ $\frac{dy}{dx} = y$

Tabulate $y(x)$ with

- The equation $y = Ce^{x}$ represents the solution curve, so duplicate the first line plot inside the graph details by selecting and performing a **`Copy/Paste`**. Replace the ***y*** in the line description with Ce^{x} and change the color of the line.
- You must give ***C*** a value before this plot will show. In a new Prop, input **`1`** as this value. **`Clarify`** and declare ***C*** a constant. The plot will generate in the Graph Theory.
- To better display the Euler line, generate a scatter plot of the points. Select the table again and choose **`Add Scatter Plot`** under the **`Graph`** ▶ **`Additional`** menu. *MathView* will automatically add the 21 scatter points over the Euler line. Displayed below is the resulting graph theory.



☐ $\frac{dy}{dx} = y$

Tabulate $y(x)$ with

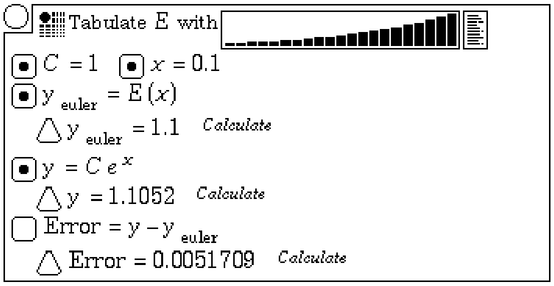
● $C = 1$

The Euler curve is a pretty good approximation close to the starting point, but contains more error the farther right you go. To analyze the error, the following Case Theory can help.

- Copy the original table and paste inside a separate Case Theory. Rename the table E (**`User Defined Function`**). Define ***C*=1** and ***x*=0.1**.
- Set up two equations: the first, defining the actual solution $y = Ce^{x}$; and the second, as the table $E(x)$. Call this second equation y_{euler}.
- Finally, create an error equation, defined as the difference, named **Error** . After calculating, **`Substitute`** the two conclusions into the **Error** equation to determine the error. You can now change the value of ***x*** to see different errors

Declare Error and euler User Defined Variables.

for different values of ***x***.



Tabulate E with

☑ $C = 1$ ☑ $x = 0.1$

☑ $y_{\text{euler}} = E(x)$

△ $y_{\text{euler}} = 1.1$ Calculate

☑ $y = C e^{x}$

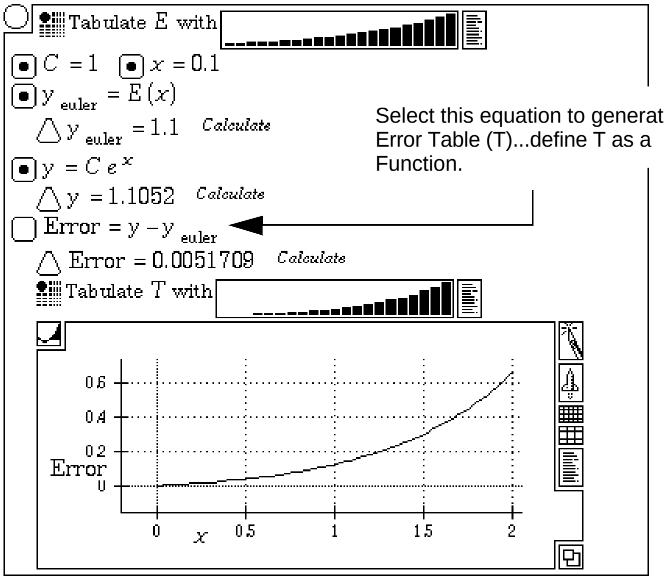
△ $y = 1.1052$ Calculate

☐ $\text{Error} = y - y_{\text{euler}}$

△ $\text{Error} = 0.0051709$ Calculate

You can further study the errors by generating another table.

- Select the Error equation and generate a table with the same data you used to generate the original Euler table. To generate the error graph, select the table and choose **Linear** under the **Graph** menu. Make ***x*** the ***x***-axis and **Error** the ***y***-axis.



Tabulate E with

☑ $C = 1$ ☑ $x = 0.1$

☑ $y_{\text{euler}} = E(x)$

△ $y_{\text{euler}} = 1.1$ Calculate

☑ $y = C e^{x}$

△ $y = 1.1052$ Calculate

☐ $\text{Error} = y - y_{\text{euler}}$

△ $\text{Error} = 0.0051709$ Calculate

Tabulate T with

Select this equation to generate the Error Table (T)...define T as a Function.

Runge-Kutta Method

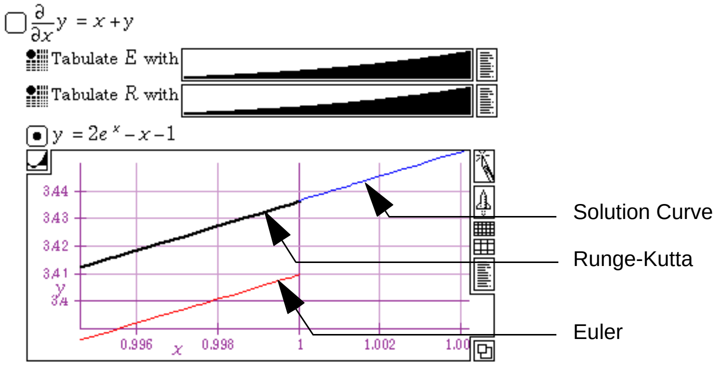
A much more accurate and more widely used method of approximation is the Runge-Kutta method. It uses a sampling of slopes through an interval and takes a weighted average to determine the right end point. This averaging gives a very accurate approximation.

- Input the following equations. The second equation is the exact solution.

☐ $\frac{\partial}{\partial x}y = x + y$ ☐ $y = 2e^{x} - x - 1$

- Generate two tables from the same differential equation, one using Euler's method and the other using the Runge-Kutta method. Make the domain 0 to 1 with 101 points. This will create an h of 0.01. Name the Euler table ***E*** and the

Runge-Kutta table ***R*** (User Defined Functions). Plot both approximations along with the solution curve in the same graph theory.



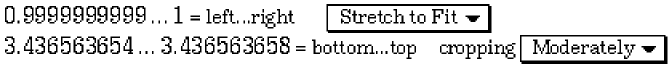
☐ $\frac{\partial}{\partial x}y = x + y$

Tabulate E with

Tabulate R with

● $y = 2e^{x} - x - 1$

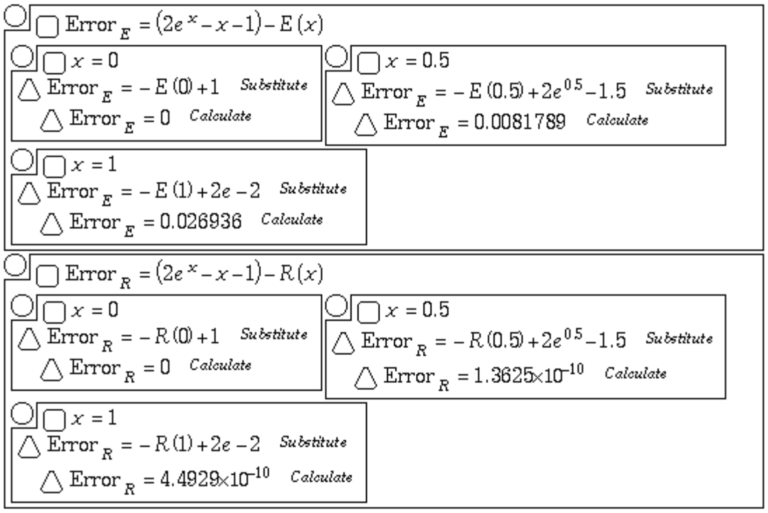
Adjust the graph bounds to show the curves at $x = 1$. The Runge-Kutta is so accurate that it appears to be on top of the actual solution curve. By zooming in on these two lines, you can generate a graph showing an error. This requires quite a few zooms. Alternatively, you can change the graph bounds to the following numbers to produce a graph showing the error.



0.9999999999 … 1 = left…right Stretch to Fit

3.436563654 … 3.436563658 = bottom…top cropping Moderately

You can further study the errors by adding the following Case Theories. Generate errors for both Euler and Runge-Kutta at $x = 0$, $x = 0.5$, and $x = 1$.



○ ☐ $\text{Error}_E = (2e^{x} - x - 1) - E(x)$

○ ☐ $x = 0$

△ $\text{Error}_E = -E(0) + 1$ *Substitute*

△ $\text{Error}_E = 0$ *Calculate*

○ ☐ $x = 0.5$

△ $\text{Error}_E = -E(0.5) + 2e^{0.5} - 1.5$ *Substitute*

△ $\text{Error}_E = 0.0081789$ *Calculate*

○ ☐ $x = 1$

△ $\text{Error}_E = -E(1) + 2e - 2$ *Substitute*

△ $\text{Error}_E = 0.026936$ *Calculate*

○ ☐ $\text{Error}_R = (2e^{x} - x - 1) - R(x)$

○ ☐ $x = 0$

△ $\text{Error}_R = -R(0) + 1$ *Substitute*

△ $\text{Error}_R = 0$ *Calculate*

○ ☐ $x = 0.5$

△ $\text{Error}_R = -R(0.5) + 2e^{0.5} - 1.5$ *Substitute*

△ $\text{Error}_R = 1.3625 \times 10^{-10}$ *Calculate*

○ ☐ $x = 1$

△ $\text{Error}_R = -R(1) + 2e - 2$ *Substitute*

△ $\text{Error}_R = 4.4929 \times 10^{-10}$ *Calculate*

Solving Second and Higher Order Equations

You solve higher order equations by forming the system of associated first order equations, selecting all by Shift-clicking, and choosing **Manipulate ▶ Table ▶ Integrate Differential Equation...**. Enter the initial values and click OK. You can generate an interesting plot from second order equations by leaving the selection as it is and choosing **Graph ▶ Other ▶ x=f(t), y=g(t) Parametric**. For higher order equations, you use **Graph ▶ Other ▶ x=f(t), y=g(t), z=h(t) Space Curve** to plot a 3-D parametric graph.