Differential Calculus

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Limits

- Tables
- Entering Greek letters
- Linear Graphs
- Dynamic graph Viewports and adding graph axes

You can use *MathView* to solve limits using algebraic manipulation, table generation, and Graph Theories. The first two methods shown in this section are conventional in their approach. The third shows you how to creatively use Graph Theories to study this sometimes difficult subject.

Limits by Algebraic Manipulation

You can find the limits of all polynomials and most rational functions using substitution. In other words, the following holds true.

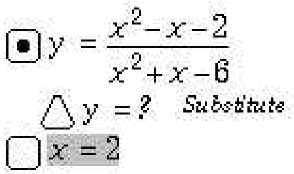
If $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$,

then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{f(c)}{g(c)}$ for $g(c) \neq 0$.

However, even when $g(c) = 0$, the limit of a rational expression may exist.

Finding the limit of the following function as $x \to 2$ poses a problem. At first look, it seems as thought the limit is undefined. The limit does exist however, and you find it by factoring the expression and simplifying.

- Input the expression as an equation of ***y*** along with a second Prop defining $x = 2$. Substituting ***x*** into the ***y*** Prop gives an undefined value, as indicated by the **?**.



● $y = \dfrac{x^2 - x - 2}{x^2 + x - 6}$

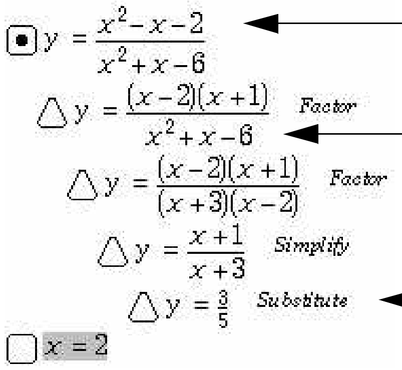
△ $y = ?$ *Substitute*

☐ $x = 2$

Value undefined because denominator is zero when **x=2**.

- **Factor** both the numerator and the denominator and **Simplify** the result.
- Complete the operation by substituting ***x*** into this new result.

When you choose **Factor**, a warning may appear stating that the manipulation may take a long time. Ignore this warning by pressing the Return key.



● $y = \dfrac{x^2 - x - 2}{x^2 + x - 6}$

△ $y = \dfrac{(x-2)(x+1)}{x^2 + x - 6}$ *Factor*

△ $y = \dfrac{(x-2)(x+1)}{(x+3)(x-2)}$ *Factor*

△ $y = \dfrac{x+1}{x+3}$ *Simplify*

△ $y = \frac{3}{5}$ *Substitute*

☐ $x = 2$

Factor numerator

Factor denominator

Limit

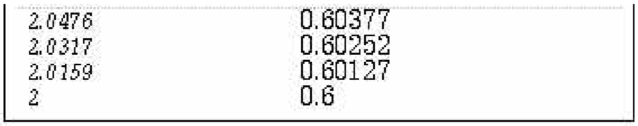
You can also use the table generator to analyze this limit. In the next example, you generate two tables. The first takes ***x*** values approaching from the left and the second takes ***x*** values approaching from the right.

Studying Limits with Tables

- Enter the expression in a new Prop, select, and generate a table defining the domain of ***x*** as 1..2. Accept the other defaults by clicking **OK** or by pressing return. Generate a second Table by selecting the expression again and defining the domain of ***x***, this time as 3..2. Open the details and observe that both tables approach .60 or 3/5, matching the results from the last example.

$$\frac{x^2-x-2}{x^2+x-6}$$

Bottom of Table #2



```
2.0476          0.60377
2.0317          0.60252
2.0159          0.60127
2               0.6
```

A Graphical look at δ–ε

You can use a Graph Theory to explore the formal definition of a limit. Below, as review, is the conventional definition.

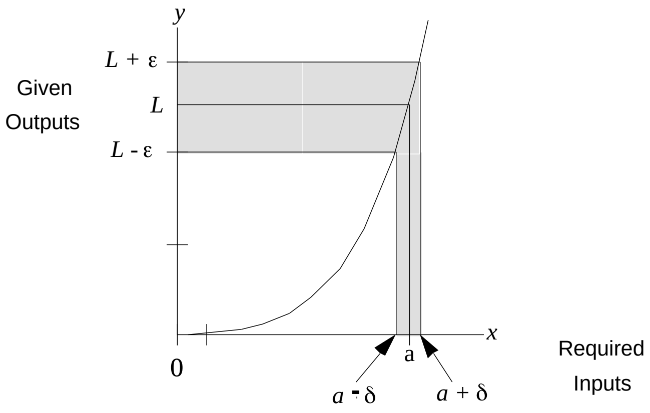
The limit $\lim_{x \to a} f(x) = L$ exists if and only if,

given any $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$

whenever $0 < |x - a| < \delta$.

Given a point $a \to f(a)$, δ is the amount of input error accepted given an output error of ε. If this relationship holds, then a limit exists and is equal to ***L***.

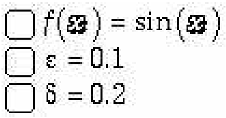
Think of it this way. Given a positive output error ε, what value of the input error δ gives an output that is within $f(a) \pm \varepsilon$?



The ability to control the Viewport of Graph Theories allows you to determine the value for which δ produces output values $f(\delta)$ that are within $L+\varepsilon$ and $L-\varepsilon$. You do this by adjusting the value of δ and keeping ε constant so the plot makes a diagonal within the graph bounds. With the values obtained from this procedure, you can find a relationship between δ and ε which closely approximates the one achieved when using traditional methods.

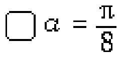
You can obtain both δ (delta) and ε (epsilon) by selecting them from the Greek Pop-up menu on the Palette.

- Use a New Notebook. Enter, in functional form with Wildcards (page 64), the sine function. In separate Props below the function, enter values for the desired output tolerance ε and a guess for the input tolerance δ. Define both as User Defined variables, when asked.



☐ $f(⊠) = \sin(⊠)$
☐ $\varepsilon = 0.1$
☐ $\delta = 0.2$

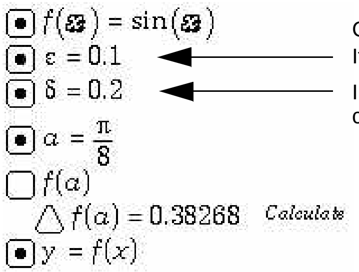
- Below these Props, input the point ***a*** defining the desired ***x*** value.



☐ $a = \frac{\pi}{8}$

Although not necessary for the plot, you may want to determine $f(a)$ by entering it in a separate Prop and performing a **Calculate**.

- Finally, generate the plot by inputting $y = f(x)$ and choosing **Linear** from the **Graph** menu. The following graphic shows the input thus far without the plot.



● $f(⊠) = \sin(⊠)$
● $\varepsilon = 0.1$
● $\delta = 0.2$
● $a = \frac{\pi}{8}$
☐ $f(a)$
△ $f(a) = 0.38268$ Calculate
● $y = f(x)$

Chosen Output error
It stays constant

Input error that you
change to make the plot move

Before the graph can help determine the correct values, you must place the Viewport details in a form representing the limits of the tolerance values.

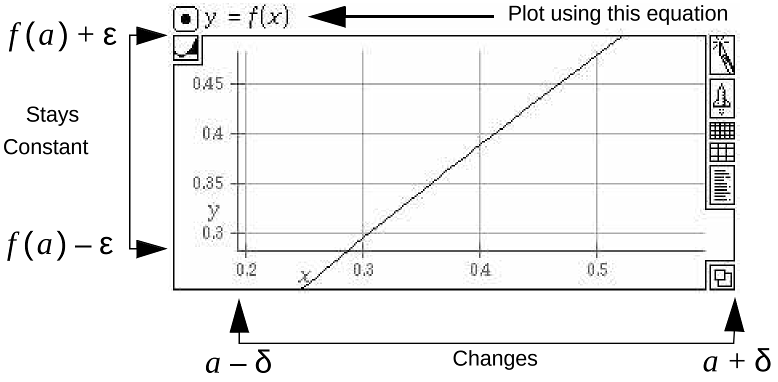
- Open the Viewport by clicking on the Graph details icon and change the ranges to the values below.

Add a small amount to ***a*** so *MathView* can generate the plot where ***x*** may be at a discontinuity, or where the limit possibly does not exist. In most cases, a small amount added to ***a*** will not affect the results.

$a - \delta \ldots a + \delta$ = left...right Stretch to Fit
$f(a) - \varepsilon \ldots f(a) + \varepsilon$ = bottom...top cropping Moderately

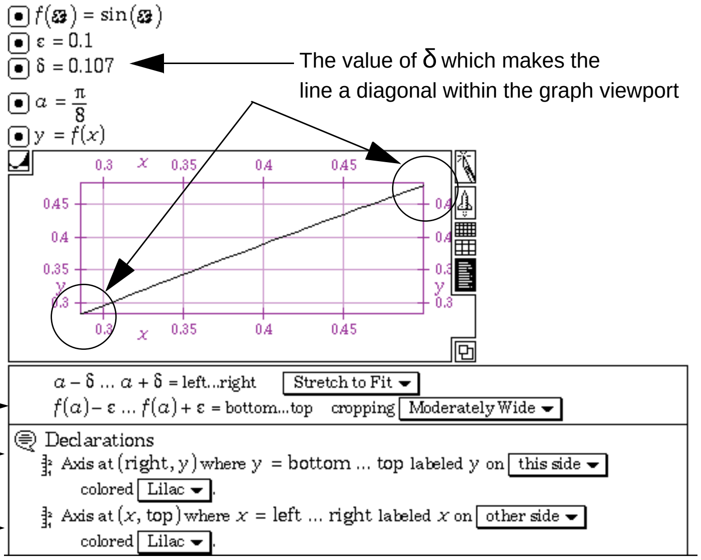
- Below is the resulting Graph Theory.

If the graph icons are in the way of the upper right corner, add one axis to the right and the other to the top of the graph. Change the cropping to Moderately Wide.



- Keep ε constant at 0.1 and change δ until the line in the plot creates a diagonal within the Viewport.

The value given to δ to create the diagonal represents the *maximum* δ can be to have an output within the limits defined by ε. The following plot suggests an approximate maximum value of 0.107.


● $f(x) = \sin(x)$
● $ε = 0.1$
● $δ = 0.107$
● $a = \frac{\pi}{8}$
● $y = f(x)$

$a - δ \ldots a + δ$ = left...right Stretch to Fit
$f(a) - ε \ldots f(a) + ε$ = bottom...top cropping Moderately Wide

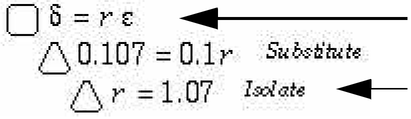
Declarations
Axis at (right, y) where y = bottom ... top labeled y on this side colored Lilac.
Axis at (x, top) where x = left ... right labeled x on other side colored Lilac.

Change the cropping.

Choose **Graph ▶ Additional ▶ Add Axis** and change the labels to those shown.

To find the relationship between δ and ε, set up a Prop creating a direct relationship.

- **Substitute** ε and δ into this new Prop.
- **Isolate *r*** (declare ***r*** a User Defined variable).
- **Substitute** the $r = 1.07$ equation back into the first Prop (see below) to give the relationship.


☐ $δ = r\,ε$
△ $0.107 = 0.1r$ Substitute
△ $r = 1.07$ Isolate

Substitute δ and ε from above into this Prop.

Substitute this equation back into the first Prop.

- Substituting ***r*** back into the first Prop gives the relationship between δ and ε.

△ $\delta = 1.07\,\varepsilon$ *Substitute*

- Below is the whole Theory.

Declarations

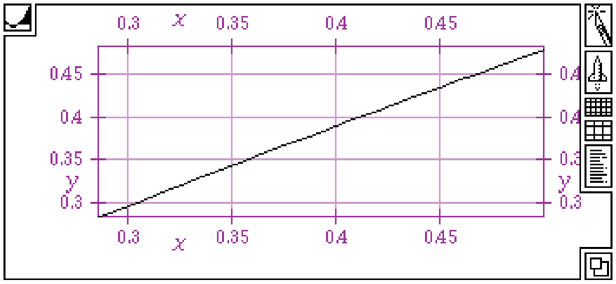
● $f(\otimes) = \sin(\otimes)$

● $\varepsilon = 0.1$

● $\delta = 0.107$

● $a = \frac{\pi}{8}$

● $y = f(x)$



☐ $\delta = r\,\varepsilon$

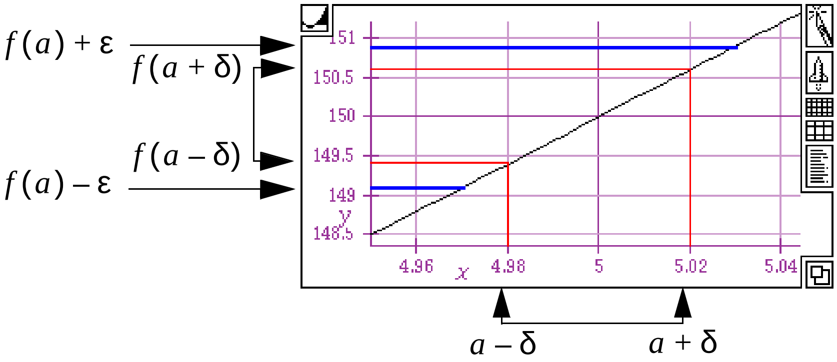
△ $0.107 = 0.1r$ *Substitute*

△ $r = 1.07$ *Isolate*

△ $\delta = 1.07\,\varepsilon$ *Substitute*

Teacher's Note

You can add line plots, defining the limits, to the Graph Theory. As different values are input, the lines act as the bounds, rather than the Viewport. The Theory below shows an example.



Slope of a Curve

- Zooming in and out on a plot
- Adding line plots
- Adjusting Viewports
- Animation
- Secant and Tangent lines
- Taylor Series

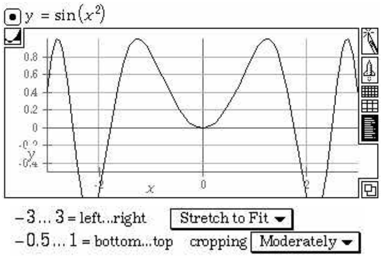
The derivative of a function is inexorably linked to the concept of the slope of a curve at a given point. An efficient way to study this concept is to use Graph Theories. This section demonstrates how to zoom in on a curve to approximate slope; how to add a line plot to a Graph Theory to represent a secant to a given function; and how to make tangent lines using function notation. Finally, you are introduced to the Taylor Series manipulation by using the feature to generate the same tangent line.

Zooming in on Curves

A change in ***y*** (the outputs), divided by the associated change in ***x*** (the inputs), determines the slope of a straight line. A problem occurs, however, when you try this method with curves. For any change in ***x***, the associated change in ***y*** determines the average change, or the slope of the *secant line*, between the two points.

By taking smaller and smaller changes in ***x***, this value approaches the true slope of the curve. Zooming in on a curve is a graphical way you can display this concept. After you zoom in on a plot several times, the curve will look more and more like that of a straight line and you can calculate an approximation to the actual slope of the curve at a given point.

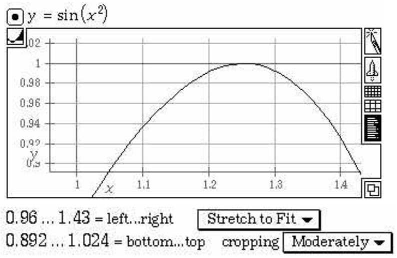
- Input the following function and zoom by using the Knife, using the Rocket (hold down the Option or Alt key while clicking on the Rocket), or opening the details and changing the Viewport. These details are shown below each of the following graphs. The chosen point is $x = 1.2$. The first graph shows the curve as *MathView* initially generates it. The second, and subsequent graphs, show various zooms.



● $y = \sin(x^2)$

$-3 \ldots 3$ = left...right Stretch to Fit

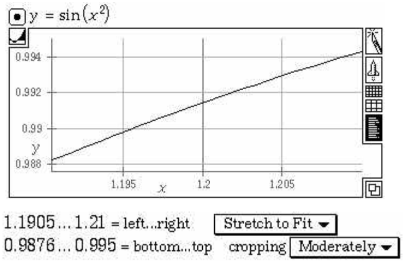
$-0.5 \ldots 1$ = bottom...top cropping Moderately



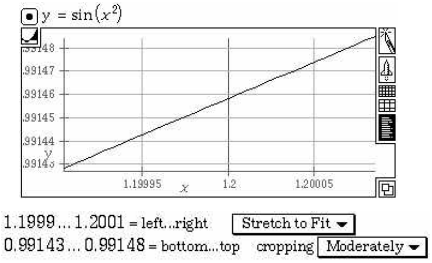
● $y = \sin(x^2)$

$0.96 \ldots 1.43$ = left...right Stretch to Fit

$0.892 \ldots 1.024$ = bottom...top cropping Moderately



1.1905 ... 1.21 = left...right
0.9876 ... 0.995 = bottom...top

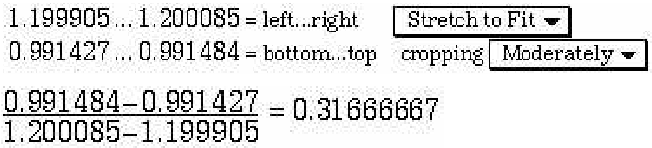


1.1999 ... 1.2001 = left...right
0.99143 ... 0.99148 = bottom...top

Although the line does not intersect a gridline exactly, you can approximate the plot's slope by using the ***x*** values of 1.19995 and 1.20005 in the last graph. The change in the associated ***y*** values divided by the differences in the ***x*** values gives a slope of .30. This number is pretty close to the actual slope of .31302.

You can also use the Viewport to determine the slope. Zoom in to a view where the line makes a diagonal in the Viewport. Then use the Viewport boundary values to perform the calculations. This calculation leads to a slope of .3166...

Change the precision to 8 by choosing **Notebook ▶ Display Precision ▶ 8 digits**.



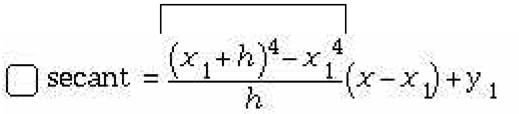
1.199905 ... 1.200085 = left...right
0.991427 ... 0.991484 = bottom...top

$$\frac{0.991484-0.991427}{1.200085-1.199905} = 0.31666667$$

Secant Lines

In the last example, you determined the slope of the secant line by connecting two points near a target point. In the next example, you will add a secant line to the plot of a function using the difference quotient $[f(x+h)-f(x)]/h$. In addition, by using *MathView's* animation feature, you will turn that secant line into a tangent line.

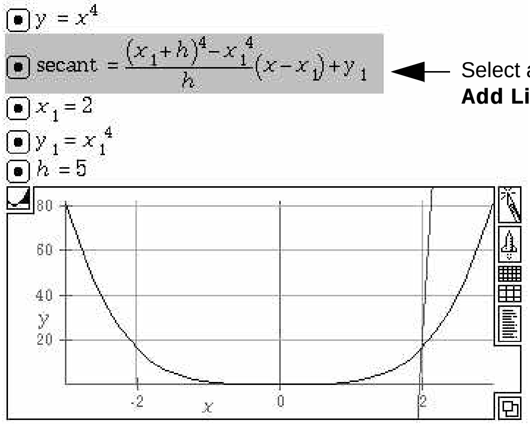
- In a New Notebook, input the function $y = x^4$. Generate a Linear plot and press return. This places you in a Prop between the function and its graph. Input the equation for the secant in Point-slope form using subscripted variables.



Difference Quotient ***m***

$$\square \text{ secant} = \frac{(x_1+h)^4 - x_1^4}{h}(x - x_1) + y_1$$

- Input a value of 2 for x_1 and define y_1 to be x_1^4. This input allows you to change x_1 without having to determine y_1 for any given point. Finally, create a Prop defining $h = 5$. Declare all new names User Defined when requested by *MathView.*
- Select the ***secant*** Prop and choose **Graph ▶ Additional ▶ Add Line Plot**. The notebook will now look like the following.



● $y = x^4$

● $\text{secant} = \frac{(x_1+h)^4 - x_1^4}{h}(x - x_1) + y_1$

● $x_1 = 2$

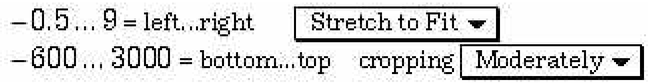
● $y_1 = x_1^4$

● $h = 5$

Select and choose **Add Line Plot**.

- Change the Viewport ranges to better display the plot.

Make sure you define the ***x***-axis and the ***y***-axis variables as ***x*** and ***secant*** , respectively.

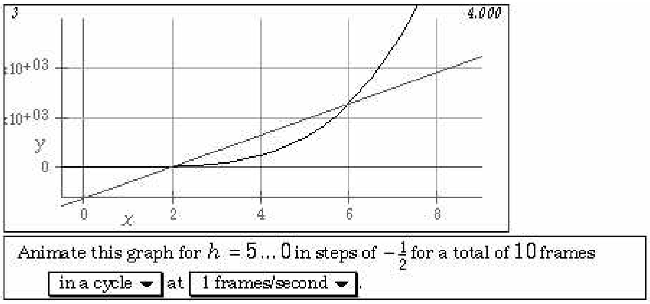


$-0.5 \ldots 9$ = left...right   Stretch to Fit ▾

$-600 \ldots 3000$ = bottom...top   cropping Moderately ▾

Change ***h*** from 5 → 0 and the secant will rotate towards a tangent line at 2.

- You can automate this by selecting ***h***, or its equation, and choosing **Graph ▶ Animation ▶ Start**. Change the animation range from the default of $0 \quad 2\pi$ to a range defined as $5 \quad 0$. The screen-shot below has caught the animation where $h = 4.0$.



Animate this graph for $h = 5 \ldots 0$ in steps of $-\frac{1}{2}$ for a total of 10 frames in a cycle ▾ at 1 frames/second ▾.

Tangent Lines

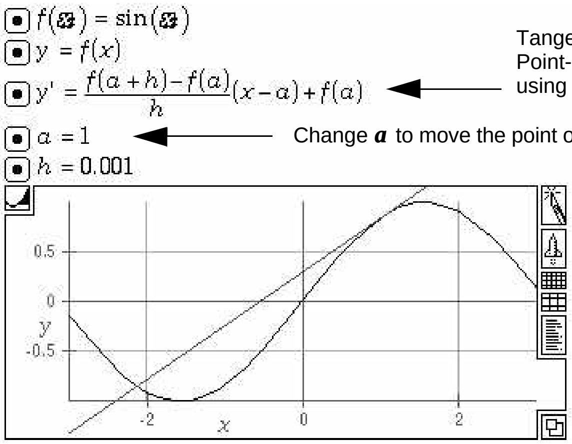
In the next example, you will use function notation to plot a tangent line at a point ***a*** on the Sine curve. When you change the value of ***a***, the tangent will move to different points on the curve.

You may want to try the method described here for the Secant Line example above.

- In a New Notebook, or in a Case Theory within an existing notebook, input the sine function using Wildcard variables. Plot this function using a new Prop where $y = f(x)$. Declare ***f*** as a function.
- After the plot generates, hit the Return key. Enter the equation for the tangent, using function notation. See the y' Prop below.
- Create two additional Props defining $a = 1$ and $h = 0.001$. Select the Prop defining the tangent line and **Add Line Plot**, making sure to choose the correct

x and ***y*** axis variables (x and y'). The Theory will look like the following, after declaring all variables User Defined.

By zooming out and animating ***a***, watch the line traverse the sine wave like a skier going over the bumps on a ski slope.



● $f(\circledast) = \sin(\circledast)$

● $y = f(x)$

● $y' = \dfrac{f(a+h)-f(a)}{h}(x-a)+f(a)$

● $a = 1$

● $h = 0.001$

Tangent line in Point-Slope form using ***a*** as the point.

Change ***a*** to move the point of tangency.

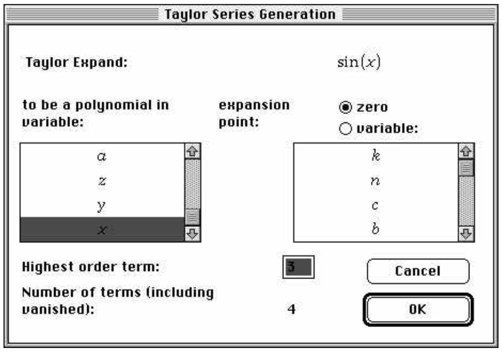
Linearization using Taylor Series

Using the same function, the next example demonstrates *MathView*'s Taylor Series. The tangent line is, after all, just a Taylor polynomial of degree one.

- Input the sine function in a new Prop within a Case Theory, or in a New Notebook. Do not use Wildcard variables this time. Select the function and generate a Linear Graph Theory.

☐ $y = \sin(x)$

- Select the RHS (right hand side of the equation) and choose **Manipulate** ▶ **Other** ▶ **Taylor Series**. The following dialog box will open.



Taylor Expand: $\sin(x)$

to be a polynomial in variable: a, z, y, x (selected: x)

expansion point: ● zero ○ variable: k, n, c, b

Highest order term: 3

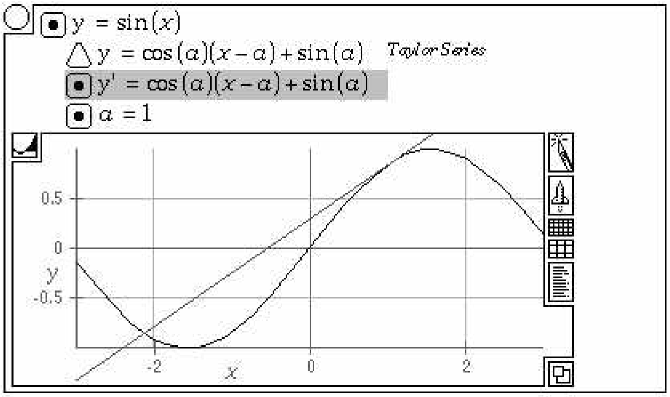
Number of terms (including vanished): 4

Since you want to have the ability to move the tangent line, choose an expansion variable.

- Toggle on the variable button by clicking on it, and choose ***a*** in the scrollable box on the right. Change the **Highest order term** to 1 and click **OK**, or press return. *MathView* generates a new equation with the RHS (right hand side of the equation) in the form of a Taylor series.
- Press return again and input a Prop giving ***a*** the value of 1.

Rename the equation at this time. This action allows you to plot the Taylor equation in the same Graph Theory without conflicting with the other plot (the existing plot already uses ***y*** as the dependent variable).

- Select the ***y*** in the Taylor equation and type y' . *MathView* generates a new Assumption just below the Taylor equation. Select this new equation and choose **Graph ▶ Additional ▶ Add Line Plot**. A Graph Theory exactly like the one in the last example appears.



● $y = \sin(x)$

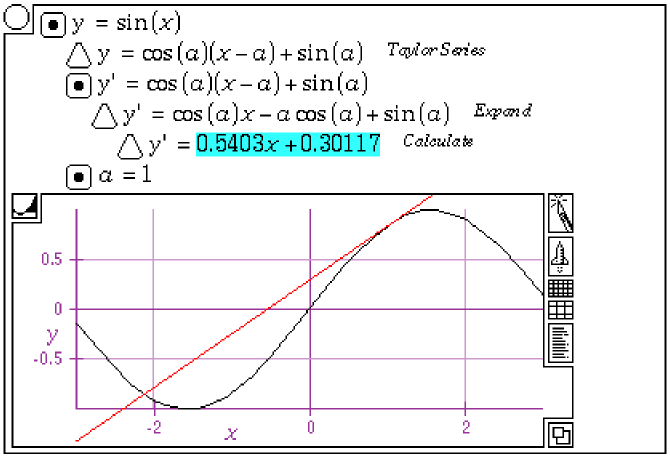
△ $y = \cos(a)(x-a)+\sin(a)$ Taylor Series

● $y' = \cos(a)(x-a)+\sin(a)$

● $a = 1$

- Change the value of ***a*** to move the line. You can also animate this graph by choosing ***a*** as the animation variable.

The Taylor series does not look very much like a polynomial at this point because you have not substituted the expansion variable into the equation yet. To see the series in polynomial form, substitute ***a*=1** into the equation, **Expand**, and then **Calculate** the RHS; or since you have given ***a*** a value in the Theory, just select the RHS, **Expand**, and **Calculate**.



● $y = \sin(x)$

△ $y = \cos(a)(x-a)+\sin(a)$ Taylor Series

● $y' = \cos(a)(x-a)+\sin(a)$

△ $y' = \cos(a)x - a\cos(a)+\sin(a)$ Expand

△ $y' = 0.5403x + 0.30117$ Calculate

● $a = 1$

Derivatives of Functions with One Variable

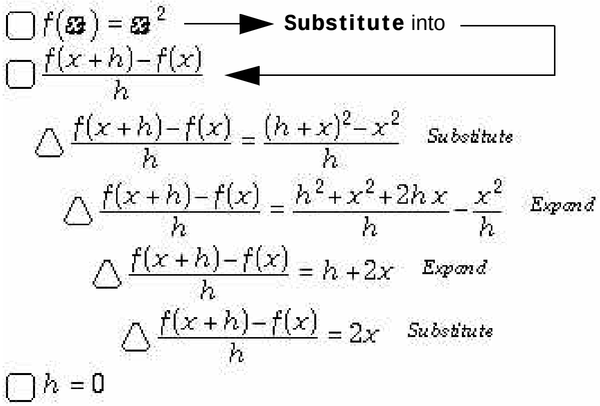
- Partial Derivative Op
- Command line derivative entry
- Numeric derivatives
- Pre-defined Differential Op

You can find derivatives of functions with one variable using two different methods in *MathView*. The preferred method is to use the Partial Derivative Op. A second method is to use the PreDefined Differential Operator, ***d***. Before the discussion on these operators, you will explore the link between the Difference Quotient and the derivative, giving you another look at how you can use functions in *MathView.*

Solving the Difference Quotient

In the next example, you algebraically solve the difference quotient using the simple function $f(x) = x^2$. The solution, in turn, gives the derivative of the function.

- Input the function using Wildcard variables.
- In a second Prop, input the difference quotient using regular variables.
- Select the function and substitute it into the difference quotient Prop. After you declare the function ***f***, *MathView* performs the operation.
- Select the RHS of the resulting equation and **`Expand`** twice.
- Since you are interested in the limit of this function as $h \to 0$, create a Prop defining $h = 0$ and substitute *h* into the RHS of the last equation for the answer.



$$f(\boxtimes) = \boxtimes^2$$

$$\frac{f(x+h)-f(x)}{h}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{(h+x)^2-x^2}{h}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{h^2+x^2+2hx}{h}-\frac{x^2}{h}$$

$$\frac{f(x+h)-f(x)}{h} = h+2x$$

$$\frac{f(x+h)-f(x)}{h} = 2x$$

$$h = 0$$

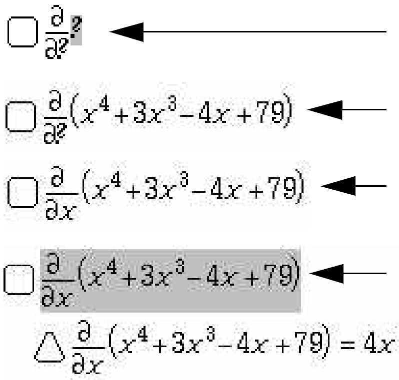
Using the Partial Derivative Op

For most of your work with derivatives, use the Partial Derivative Op. *MathView* often requires you to declare an Independence Declaration when performing partial differentiation. Therefore, as long as you do not declare an Independence Declaration, you can use the Op to take regular derivatives. See page 165 for instruction on Partial Derivatives.

You have several methods of entering the Partial Derivative Op. The first method is to click on the Palette image, which inserts the Op at your insertion point, or in a new Prop if you have no selection. Question mark place-holders await the input of the parameters.

- Click on the palette image of the Derivative Op.
- Type the expression delimited with a parenthesis and press tab to input the independent variable. Once you have entered the expression, select and **`Simplify`** to obtain a symbolic answer. Below is an example.

If you do not use parentheses to delimit the function, *MathView* will only take the derivative of the first term, x^4, and you will end up with the wrong answer: $7x^3 - 4x + 79$.



Input Partial Derivative Op.

Type the function in parentheses.

Tab and type an **x**.

Select and **`Simplify`**.

☐ $\frac{\partial}{\partial ?}?$

☐ $\frac{\partial}{\partial ?}(x^4+3x^3-4x+79)$

☐ $\frac{\partial}{\partial x}(x^4+3x^3-4x+79)$

☐ $\frac{\partial}{\partial x}(x^4+3x^3-4x+79)$

△ $\frac{\partial}{\partial x}(x^4+3x^3-4x+79) = 4x^3+9x^2-4$ *Simplify*

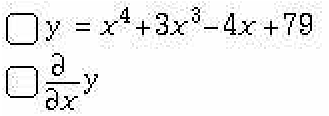
The second method involves entering the expression, selecting it, and clicking on the Palette image of the Op. Tab to input the independent variable.

The third method involves typing the command line:

```
Diff(x) esc *(x^4+3*x^3-4*x+7.9
```

A more dynamic way of working with derivatives is to create a function ***y*** and substitute it into the derivative Op.

- Input the function and the Derivative Op using ***y*** as the dependent variable and ***x*** as the independent variable.

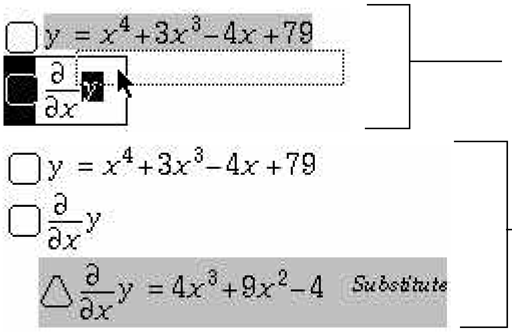


☐ $y = x^4+3x^3-4x+79$

☐ $\frac{\partial}{\partial x}y$

- Select the equation by clicking on its equal sign and, with the hand, move it to the Derivative Op. Let go.

You can now change the original function and *MathView* will automatically generate the derivative for the new function.



Manipulation

Result

☐ $y = x^4+3x^3-4x+79$

☐ $\frac{\partial}{\partial x}y$

☐ $y = x^4+3x^3-4x+79$

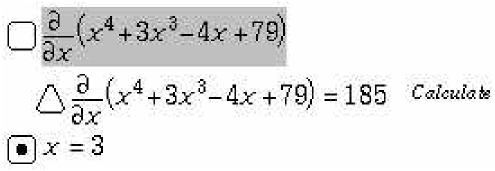
☐ $\frac{\partial}{\partial x}y$

△ $\frac{\partial}{\partial x}y = 4x^3+9x^2-4$ *Substitute*

Numeric Derivatives

All you do to generate the numeric derivative of an expression at a point is to give the independent variable a value.

- To the example above, add a Prop giving ***x*** the value of 3. Select the derivative expression and perform a **`Calculate`**.



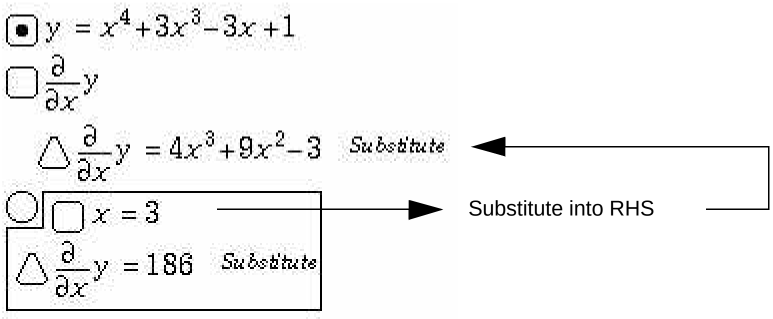
☐ $\frac{\partial}{\partial x}(x^4+3x^3-4x+79)$

△ $\frac{\partial}{\partial x}(x^4+3x^3-4x+79) = 185$ *Calculate*

⊡ $x = 3$

A problem may occur if you do not place the $x = 3$ Prop in its own Case Theory. Since you have given ***x*** a value, all subsequent derivatives will automatically result in a numerical answer. Using a Case Theory solves this problem.

- Create a Case Theory and enter $x = 3$. Substitute this equation into the Derivative Op located outside of the Case Theory. The answer displays inside the Case Theory and the value you gave to ***x*** will not affect any other derivatives.



⊡ $y = x^4+3x^3-3x+1$

☐ $\frac{\partial}{\partial x}y$

△ $\frac{\partial}{\partial x}y = 4x^3+9x^2-3$ *Substitute*

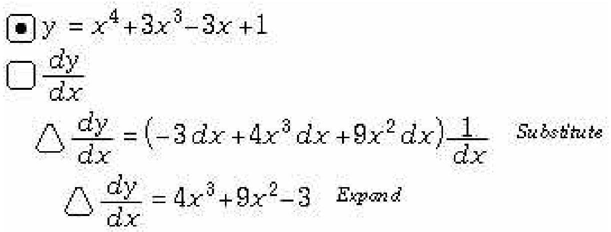
○ ☐ $x = 3$

△ $\frac{\partial}{\partial x}y = 186$ *Substitute*

Using the Differential Operator *d*

You can also use the PreDefined differential operator ***d*** to find derivatives.

- Using the same function as above, enter the derivative using the differential operator ***d***, rather than the Partial Derivative Op. **`/ d*x`** tab **`d*y`**
- Substitute the function into the derivative for the answer. The resulting expression includes ***dx*** terms, which will cancel when you perform an **`Expand`** on the RHS.



⊡ $y = x^4+3x^3-3x+1$

☐ $\frac{dy}{dx}$

△ $\frac{dy}{dx} = (-3\,dx+4x^3\,dx+9x^2\,dx)\frac{1}{dx}$ *Substitute*

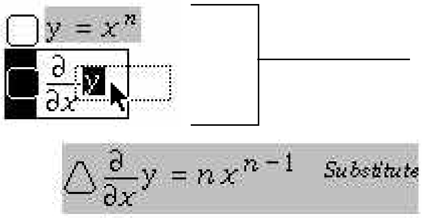
△ $\frac{dy}{dx} = 4x^3+9x^2-3$ *Expand*

Rules of Differentiation

You can explore the rules of differentiation by using substitution variables or constants, as the case requires, to represent the inner functions and powers that make up a function.

You must declare the index ***n*** a constant.

- For example, to explore the Power Rule, input the following two Props and perform the substitution. Notice that the derivative is of the function ***y*** with respect to ***x***.



$y = x^n$

$\frac{\partial}{\partial x} y$

$\frac{\partial}{\partial x} y = n x^{n-1}$ Substitute

Manipulation

Result

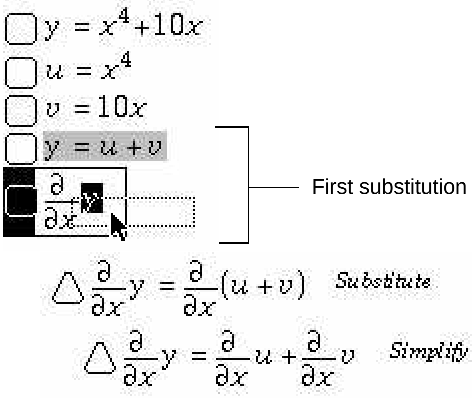
You explore the Negative Power Rule and Constant Power Rule in like fashion.

Use a variable in place of the inner functions to demonstrate several other rules of differentiation.

Make sure you declare ***u*** and ***v*** as variables.

- Below is the Sum Rule. Enter the first five Props shown below and perform the first substitution. Turn off **Auto Simplify** to have *MathView* show all of its steps.

Manipulation #1



$y = x^4+10x$

$u = x^4$

$v = 10x$

$y = u+v$

$\frac{\partial}{\partial x} y$

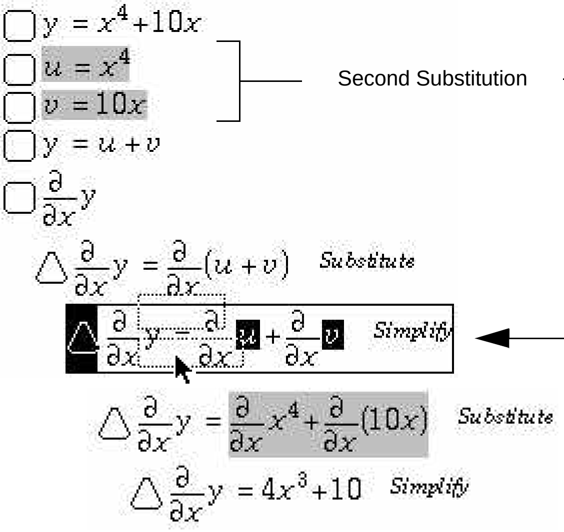
$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}(u+v)$ Substitute

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} u + \frac{\partial}{\partial x} v$ Simplify

First substitution

Result

Manipulation #2



$y = x^4+10x$

$u = x^4$

$v = 10x$

$y = u+v$

$\frac{\partial}{\partial x} y$

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}(u+v)$ Substitute

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} u + \frac{\partial}{\partial x} v$ Simplify

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} x^4 + \frac{\partial}{\partial x}(10x)$ Substitute

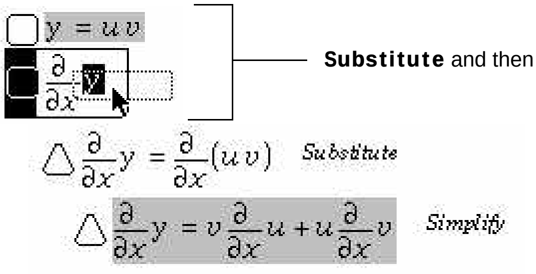
$\frac{\partial}{\partial x} y = 4x^3+10$ Simplify

Second Substitution

Select RHS and **Simplify**

You can display the Product and Quotient rules in similar fashion. The screen-shot below only shows the manipulations on the substitution variables.

Product Rule



$y = u v$

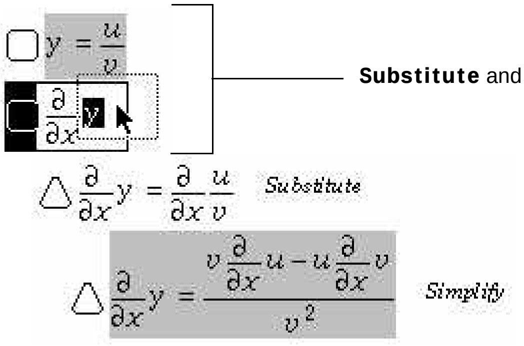
$\frac{\partial}{\partial x} y$

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x}(u v)$ Substitute

$\frac{\partial}{\partial x} y = v \frac{\partial}{\partial x} u + u \frac{\partial}{\partial x} v$ Simplify

Substitute and then **Simplify**.

Quotient Rule



$y = \frac{u}{v}$

$\frac{\partial}{\partial x} y$

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} \frac{u}{v}$ Substitute

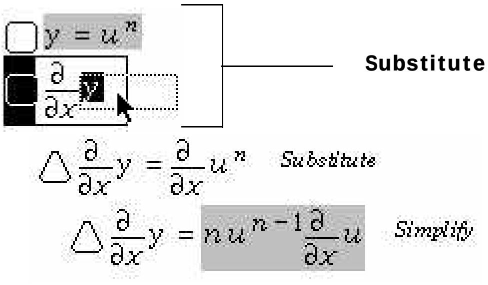
$\frac{\partial}{\partial x} y = \frac{v \frac{\partial}{\partial x} u - u \frac{\partial}{\partial x} v}{v^2}$ Simplify

Substitute and then **Simplify**.

- Use the next two examples to demonstrate the *chain rule*. Remember to declare ***n*** as a constant.

Turn off **Auto Simplify** for this manipulation. The example will then show the intermediate step.

Chain Rule



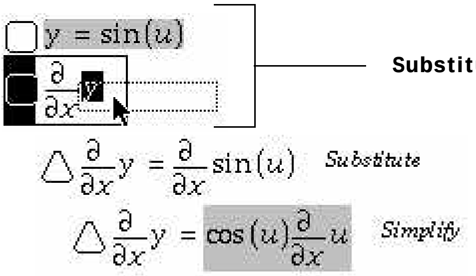
$y = u^n$

$\frac{\partial}{\partial x} y$

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} u^n$ Substitute

$\frac{\partial}{\partial x} y = n u^{n-1} \frac{\partial}{\partial x} u$ Simplify

Substitute and then **Simplify**.



$y = \sin(u)$

$\frac{\partial}{\partial x} y$

$\frac{\partial}{\partial x} y = \frac{\partial}{\partial x} \sin(u)$ Substitute

$\frac{\partial}{\partial x} y = \cos(u) \frac{\partial}{\partial x} u$ Simplify

Substitute and then **Simplify**.

Higher Order Derivatives

- Partial Derivative Op
- Linear Graph Theories
- Changing graph line details
- Inputting higher order derivatives

Not only are higher-order derivatives easy to input and solve, the dynamic aspect of *MathView* allows you to change initial functions and witness immediate changes to a Graph Theory containing their plots.

In the following example, you find the first and second derivatives of a function. You then plot all three in the same Graph Theory. By varying the initial function, you can watch all three plots change in the Graph Theory.

- Using the Partial Derivative Op, you enter the second derivative in the following manner. Click on the Palette image of the Op twice, type ***y***, press tab, type an ***x***, press tab again, and type another ***x***.

☐ $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}y\right)$ Press $\frac{\partial}{\partial a}$ $\frac{\partial}{\partial a}$ **y** tab **x** tab **x**

You can also enter a second derivative as an exponent.

- Enter a single Partial Derivative Op in a new Prop and select it without the ***y***, by clicking once on its fraction bar.
- Click on the Superscript Palette icon or press shift - 6.
- Complete the input by typing the index number.

☐ $\frac{\partial}{\partial x}y$	Input derivative of ***y*** with respect to ***x***.	$\frac{\partial}{\partial a}$
☐ $\frac{\partial}{\partial x}y$	Select the derivative Op by clicking on the fraction bar.	
☐ $\left(\frac{\partial}{\partial x}\right)^{?}y$	Index by typing shift - 6 or using the Palette image:	a^c
☐ $\left(\frac{\partial}{\partial x}\right)^{2}y$	Type the index number (2 in this case).	

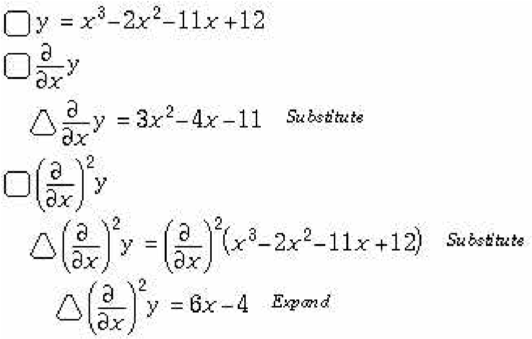
- To verify that this input is in fact taking the derivative of a derivative, select and **`Expand`**. Try using this technique with even higher order derivatives.
- Input the function and the first derivative.
- Follow by inputting the second derivative in the third Prop.
- Complete by selecting the function and **`Substituting`** it into each Derivative Op.

☐ $y = x^3 - 2x^2 - 11x + 12$ —— **`Substitute`** into

☐ $\frac{\partial}{\partial x}y$ ◀

☐ $\left(\frac{\partial}{\partial x}\right)^{2}y$ ◀

You will have to **`Expand`** the RHS of the intermediate answer of the second substitution to complete the operation.



☐ $y = x^3-2x^2-11x+12$

☐ $\frac{\partial}{\partial x}y$

△ $\frac{\partial}{\partial x}y = 3x^2-4x-11$ Substitute

☐ $\left(\frac{\partial}{\partial x}\right)^2 y$

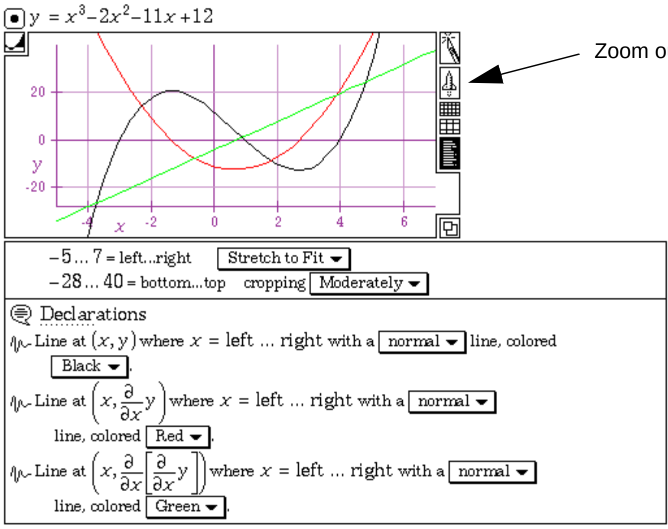
△ $\left(\frac{\partial}{\partial x}\right)^2 y = \left(\frac{\partial}{\partial x}\right)^2 \left(x^3-2x^2-11x+12\right)$ Substitute

△ $\left(\frac{\partial}{\partial x}\right)^2 y = 6x-4$ Expand

You plot these functions by selecting the original equation, generating a Linear Graph Theory, and adding two line plots. Define the first additional plot as the first derivative and the second as the second derivative.

- Select the original function and generate a Linear Graph Theory.
- Add two line plots and open the details.
- Select the ***y*** in the second line detail. Click on the Derivative Op on the Palette and type an ***x***. The Graph Theory will redraw with the derivative function, along with the original function.
- Select the ***y*** in the third line detail and click on the Derivative Op.
- Type an ***x*** and then select the whole derivative, including the ***y***, and click on the Derivative Op again.
- Complete the operation by typing another ***x***.

You can now change the original function and the affects of that change will ripple through the theory, redrawing all three plots.



● $y = x^3-2x^2-11x+12$

$-5 \ldots 7$ = left...right   Stretch to Fit

$-28 \ldots 40$ = bottom...top   cropping Moderately

Declarations

Line at (x, y) where x = left … right with a normal line, colored Black.

Line at $\left(x, \frac{\partial}{\partial x}y\right)$ where x = left … right with a normal line, colored Red.

Line at $\left(x, \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}y\right]\right)$ where x = left … right with a normal line, colored Green.

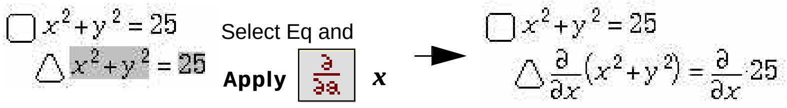
Zoom out once.

Implicit Differentiation

- Partial Derivative Op
- Apply
- Adding line plots
- Implicit Graph Theories

In this section, you learn to use the **Apply** manipulation to help differentiate implicit functions. In addition, you are introduced to the **Implicit** Graph Theory and learn how to add a line plot to one of these plots.

- Enter an equation for a circle. Select the equation and choose **Manipulate ▶ Apply**. Click on the Derivative Op palette image and type an ***x***. The apply will give you two **?** placeholders awaiting an input. The Theory will look like the following.



☐ $x^2+y^2=25$

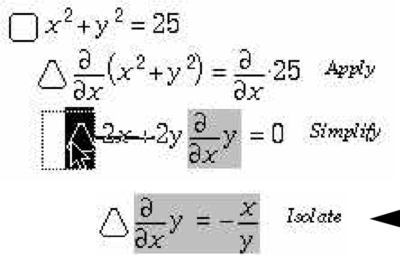
△ $x^2+y^2=25$

Select Eq and **Apply** $\frac{\partial}{\partial a}$ ***x***

☐ $x^2+y^2=25$

△ $\frac{\partial}{\partial x}(x^2+y^2)=\frac{\partial}{\partial x}\cdot 25$

- Select the equation again and **Simplify**, to perform the differentiation.
- Select the Derivative Op and **Isolate** for the answer.



☐ $x^2+y^2=25$

△ $\frac{\partial}{\partial x}(x^2+y^2)=\frac{\partial}{\partial x}\cdot 25$ *Apply*

△ $2x+2y\frac{\partial}{\partial x}y=0$ *Simplify*

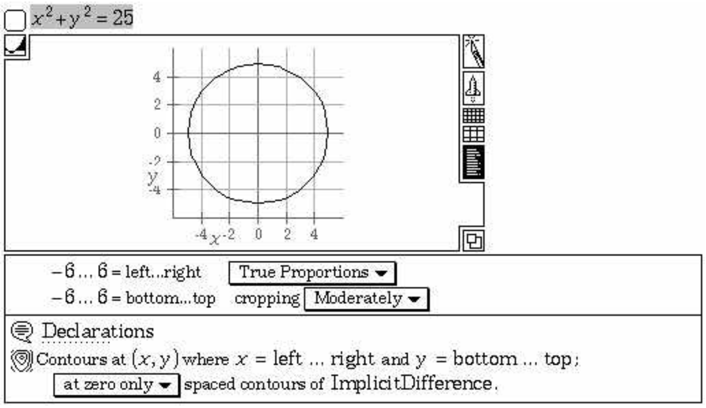
Isolate the Derivative Op.

△ $\frac{\partial}{\partial x}y=-\frac{x}{y}$ *Isolate* ◀—— Result

Implicit Graphs

- You plot an implicit equation by selecting the equation (click once on its equal sign) and choosing the implicit graph menu item, **Graph ▶ Other ▶ f(x,y) = g(x,y) Implicit**. Choose ***x*** as the ***x***-axis and ***y*** as the ***y***-axis.
- You may have to zoom out once or twice to have the circle display in the Viewport. Use **True Proportions** for this plot.

Clicking the more resolution icon will make a smoother circle.



☐ $x^2+y^2=25$

$-6\ldots6$ = left...right True Proportions ▾

$-6\ldots6$ = bottom...top cropping Moderately ▾

Declarations

Contours at (x,y) where x = left ... right and y = bottom ... top;

at zero only ▾ spaced contours of ImplicitDifference.

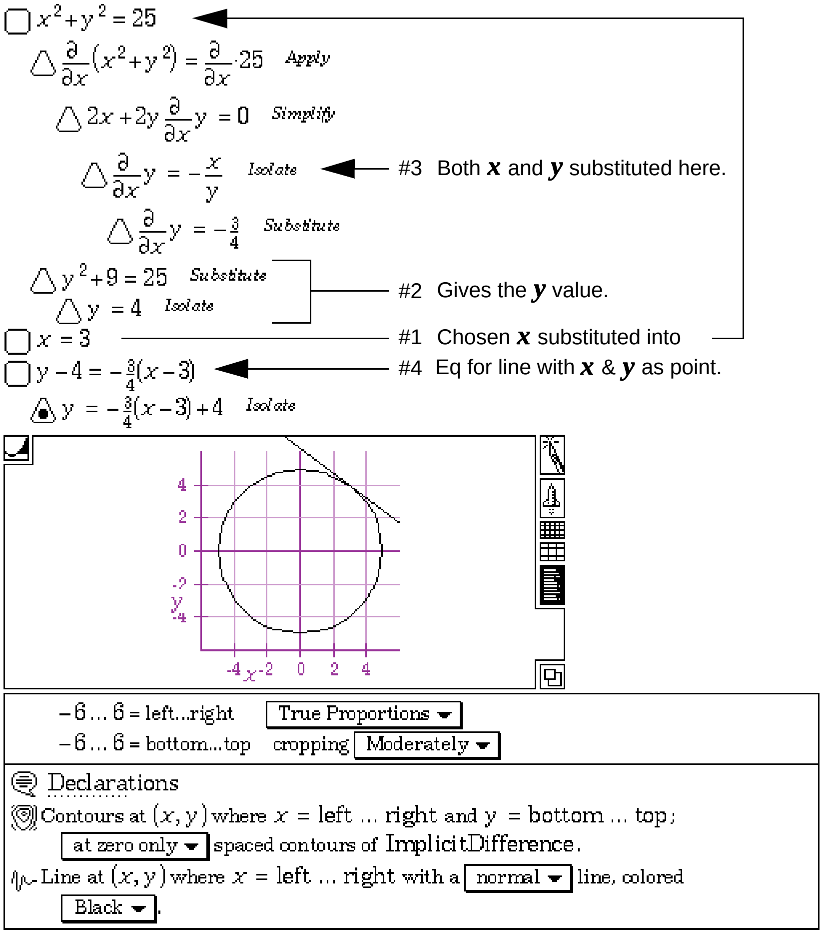
Implicit plots are merely normal Graph Theories that have zero-contour plots as their plot detail; therefore, you may add line plots.

The screen-shot below displays all of the steps in this example.

Use the point where $x = 3$.

- Select the first Prop from the previous example and press `return`. Enter $x = 3$. Substitute this equation into the original Prop and **Isolate** ***y*** (in the figure below, #1 gives #2).
- Substitute the ***x*** and ***y*** Props into the RHS of the derivative equation. This gives a slope of $-3/4$ (#3).
- Input an equation for a line into its own Prop below the $x = 3$ Prop, with the point you determined above as its inputs (#4).
- Isolate ***y*** in this equation. Add a line plot to the Graph Theory by choosing **Graph ▶ Additional ▶ Add Line Plot**. The screen-shot below displays the whole theory.

Notice that the ***y*** in the added line plot does not conflict with the ***y*** in the contour plot.



☐ $x^2+y^2=25$

△ $\frac{\partial}{\partial x}(x^2+y^2)=\frac{\partial}{\partial x}\cdot 25$ *Apply*

△ $2x+2y\frac{\partial}{\partial x}y=0$ *Simplify*

△ $\frac{\partial}{\partial x}y=-\frac{x}{y}$ *Isolate*

△ $\frac{\partial}{\partial x}y=-\frac{3}{4}$ *Substitute*

△ $y^2+9=25$ *Substitute*

△ $y=4$ *Isolate*

☐ $x=3$

☐ $y-4=-\frac{3}{4}(x-3)$

△ $y=-\frac{3}{4}(x-3)+4$ *Isolate*

#3 Both ***x*** and ***y*** substituted here.

#2 Gives the ***y*** value.

#1 Chosen ***x*** substituted into

#4 Eq for line with ***x*** & ***y*** as point.

$-6 \ldots 6$ = left...right True Proportions

$-6 \ldots 6$ = bottom...top cropping Moderately

Declarations

Contours at (x, y) where x = left … right and y = bottom … top; at zero only spaced contours of ImplicitDifference.

Line at (x, y) where x = left … right with a normal line, colored Black.

Derivatives of Functions with more than one Variable

- Case Theories
- Independence Declarations and Partial Derivative Ops
- Color 3-D plots
- Creating a tangent plane in a 3-D Graph Theory

In “Traces and Partial Derivative Functions” on page 136, you generated a graph of a function with more than one variable along with a plane intersecting that function. The plane made a curve of intersection with the three-dimensional plot called a trace. This trace represented the partial derivative with respect to the independent variable. In this section you will learn how to use *MathView* to algebraically find the derivatives of functions with more than one variable.

The point of tangency of a 3-D object is a plane. In the final example in this section you will learn how to generate this plane in a 3-D Graph Theory.

You use the Partial Derivative Op, along with an Independence Declaration, to perform partial differentiation in *MathView.*

Partial Derivatives

Use the following function for the next example.

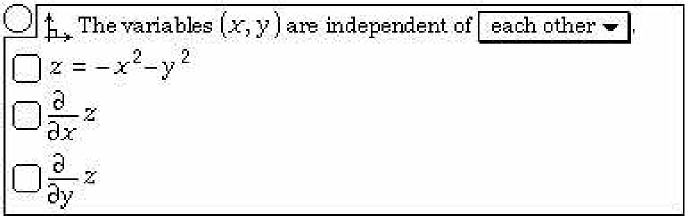
$$z = f(x, y) = -x^2 - y^2$$

Use Independence Declarations with caution, as they can adversely affect Graph Theories. *MathView* assumes the dependent variable to be a constant with respect to the independent variable and the plot produces a straight line.

- Create a Case Theory by choosing **Notebook ► Insert ► Case Theory**.

To solve partial derivatives in *MathView*, you must declare your variables independent of each other. If you put an independence declaration at the root of the notebook (outside of all Case Theories), it will affect all subsequent manipulations, including those inside Case Theories. When placed inside its own Case Theory, the declaration will only apply to operations inside that Case Theory and any Case Theories inside of it. It will not affect manipulations outside of its own Case Theory.

- Input an Independence Declaration by choosing **Notebook ► Insert ► Independence Decl.**. Declare ***x*** and ***y*** independent by typing their names separated by a coma, at the location of the **?**.
- Input the function and two Derivative Ops, the first with respect to ***x*** and the second with respect to ***y***. Your notebook will look like the following.



○ The variables (x, y) are independent of each other ▾,

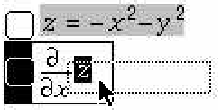
☐ $z = -x^2 - y^2$

☐ $\frac{\partial}{\partial x} z$

☐ $\frac{\partial}{\partial y} z$

To find the numerical derivative at a point is a simple matter of giving ***x*** and ***y*** values and substituting them into the two new functions just generated.

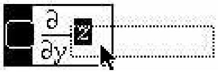
- **Substitute** the function into the derivatives to obtain the answers.



☐ $z = -x^2-y^2$

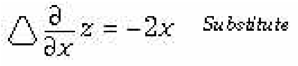
$\frac{\partial}{\partial x} z$

And

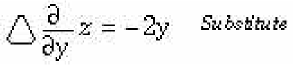


$\frac{\partial}{\partial y} z$

Results:



△ $\frac{\partial}{\partial x} z = -2x$ Substitute

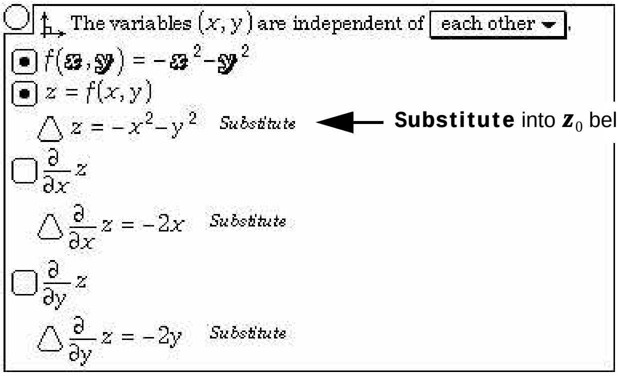


△ $\frac{\partial}{\partial y} z = -2y$ Substitute

The point of tangency of a 3-D object is a plane. The next example plots the graph of the function just used and adds a tangent plane. You can then dynamically move this plane to any point on the surface of that function.

- In a new Case Theory, input the function using Wildcard variables and enter a $z = f(x, y)$ Prop.
- Substitute the function into the ***z*** Prop and take the Partial Derivatives.

The Theory will look like the following. Do not forget the Independence Declaration, and make sure to declare ***f*** a function when requested.



○ The variables (x, y) are independent of each other.

● $f(x,y) = -x^2-y^2$

● $z = f(x,y)$

△ $z = -x^2-y^2$ Substitute

Substitute into $\boldsymbol{z}_0$ below

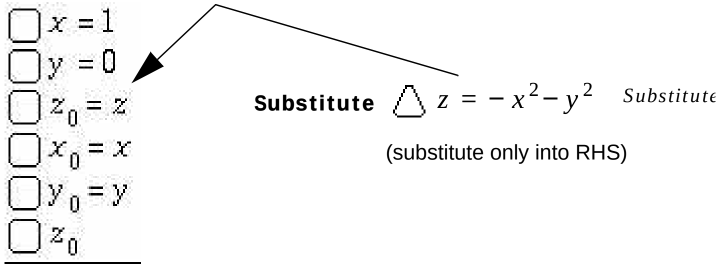
☐ $\frac{\partial}{\partial x} z$

△ $\frac{\partial}{\partial x} z = -2x$ Substitute

☐ $\frac{\partial}{\partial y} z$

△ $\frac{\partial}{\partial y} z = -2y$ Substitute

- Input the point of tangency at (1,0) by entering two Props defining ***x*** and ***y*** (still inside the Case Theory).
- Input four Props: $z_0 = z$; $x_0 = x$; $y_0 = y$; and z_0 (by itself).
- Substitute the ***z***-Prop (Conclusion), above, into the RHS of the $z_0 = z$ Prop.



☐ $x = 1$

☐ $y = 0$

☐ $z_0 = z$

☐ $x_0 = x$

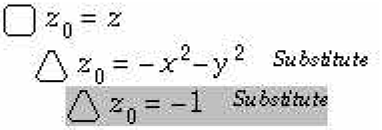
☐ $y_0 = y$

☐ z_0

Substitute △ $z = -x^2-y^2$ Substitute

(substitute only into RHS)

- Substitute the $x = 1$ and $y = 0$ Props into the resulting equation. This manipulation links the function to the plane.

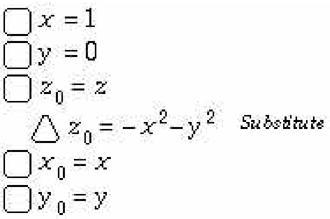


☐ $z_0 = z$

△ $z_0 = -x^2 - y^2$ Substitute

△ $z_0 = -1$ Substitute

- Substitute $x = 1$ into the RHS of the $x_0 = x$ Prop and $y = 0$ into the RHS of the $y_0 = y$ Prop.
- Finally substitute the $z_0 = -1$ Conclusion into the z_0 Prop.



☐ $x = 1$

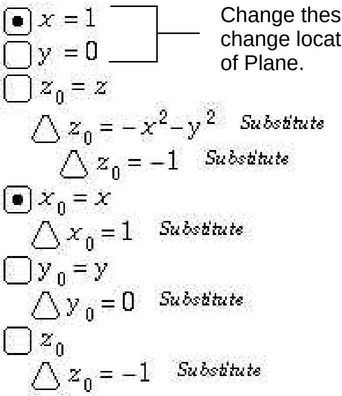
☐ $y = 0$

☐ $z_0 = z$

△ $z_0 = -x^2 - y^2$ Substitute

☐ $x_0 = x$

☐ $y_0 = y$



☑ $x = 1$

☐ $y = 0$

☐ $z_0 = z$

△ $z_0 = -x^2 - y^2$ Substitute

△ $z_0 = -1$ Substitute

☑ $x_0 = x$

△ $x_0 = 1$ Substitute

☐ $y_0 = y$

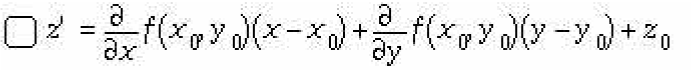
△ $y_0 = 0$ Substitute

☐ z_0

△ $z_0 = -1$ Substitute

Change these to change location of Plane.

Below is the equation for a plane.



☐ $z' = \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0) + z_0$

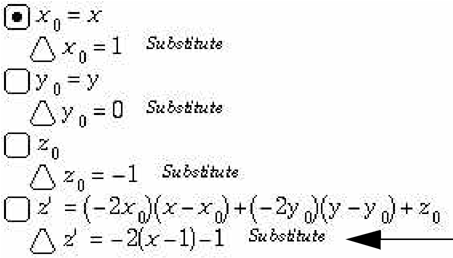
You do not use the equation above in this example, however, because the partials in the Theory are in terms of x_0 and y_0, which are constants at this time.

- You must manually input the following equation below the z_0 Prop. Declare z' as a User Defined variable when requested.

☐ $z' = (-2x_0)(x - x_0) + (-2y_0)(y - y_0) + z_0$

- Substitute the values for x_0, y_0 and z_0 in to this Prop. The resulting equation becomes the equation for the tangent plane at the point (1,0).

Make this equation the Working Statement for the plot.



☑ $x_0 = x$

△ $x_0 = 1$ Substitute

☐ $y_0 = y$

△ $y_0 = 0$ Substitute

☐ z_0

△ $z_0 = -1$ Substitute

☐ $z' = (-2x_0)(x - x_0) + (-2y_0)(y - y_0) + z_0$

△ $z' = -2(x - 1) - 1$ Substitute

Plane only has one independent variable.

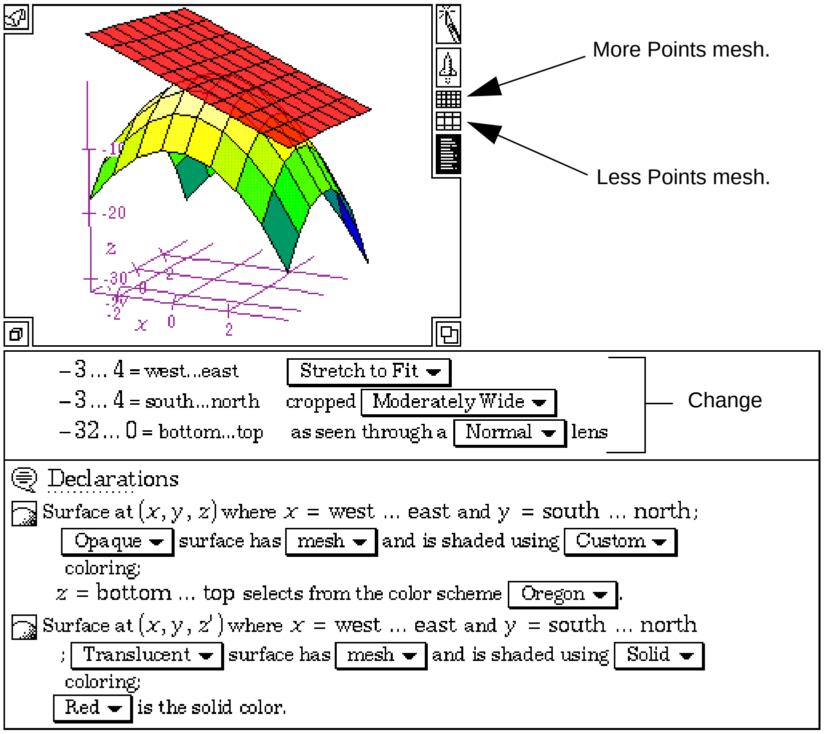
- Generate the Graph Theory by selecting the original function, the $z = f(x, y)$ Prop, and choose **Graph ▶ z=f(x,y) ▶ Color 3D**.

- Open the details and select the surface plot detail, by clicking on the leading icon and **Copy/Paste**. A second plot will generate. Change z to z' in the second detail. If the plane does not draw immediately, select the z' Conclusion (triangle Prop icon), and make it the Working Statement by selecting **Notebook ▶ Make Working Stmt**.

You use this method of adding the second plot because *MathView* needs two independent variables to add a Surface Plot, which this equation does not have. You can trick the program by adding a plot with the same definition as the first plot and then changing the detail.

Alternatively you can toggle on **Additional ▶ with blank slots** under the Graph menu. Then select the z' Prop and choose **Add Surface Plot.** This action adds a blank Surface Plot detail which you then fill in with the details shown to the right.

The screen shot below has the details open so you can generate the same plot.



More Points mesh.

Less Points mesh.

−3 … 4 = west…east Stretch to Fit

−3 … 4 = south…north cropped Moderately Wide

−32 … 0 = bottom…top as seen through a Normal lens

Change

Declarations

Surface at (x, y, z) where x = west … east and y = south … north; Opaque surface has mesh and is shaded using Custom coloring;
z = bottom … top selects from the color scheme Oregon.

Surface at (x, y, z') where x = west … east and y = south … north; Translucent surface has mesh and is shaded using Solid coloring;
Red is the solid color.

To change the location of the tangent plane, merely change the values of ***x*** and ***y*** and the Graph Theory will redraw with the plane at that new point.